

A NEW APPROACH TO VELOCITY AVERAGING LEMMAS IN BESOV SPACES

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ABSTRACT. We develop a new approach to velocity averaging lemmas based on the dispersive properties of the kinetic transport operator. This method yields unprecedented sharp results, which display, in some cases, a gain of one full derivative. Moreover, the study of dispersion allows to treat the case of $L_x^r L_v^p$ integrability with $r \leq p$. We also establish results on the control of concentrations in the degenerate $L_{x,v}^1$ case, which is fundamental in the study of the hydrodynamic limit of the Boltzmann equation.

CONTENTS

1. Introduction	1
2. Technical tools	3
2.1. Besov spaces	3
2.2. Real interpolation theory	5
3. Main results	6
3.1. Velocity averaging in $L_x^1 L_v^p$, inhomogeneous case	6
3.2. Velocity averaging in $L_x^1 L_v^p$, homogeneous case	9
3.3. The classical $L_x^2 L_v^2$ case revisited	10
3.4. The $L_x^1 L_v^p$ and $L_x^2 L_v^2$ cases reconciled	11
4. Ellipticity, dispersion and averaging	15
5. Control of concentrations in L^1	20
6. Proofs of the main results	27
6.1. Velocity averaging in $L_x^1 L_v^p$, inhomogeneous case	27
6.2. Velocity averaging in $L_x^1 L_v^p$, homogeneous case	35
6.3. The classical $L_x^2 L_v^2$ case revisited	39
6.4. The $L_x^1 L_v^p$ and $L_x^2 L_v^2$ cases reconciled	49
Appendix A. Some paradifferential calculus	59
References	62

1. INTRODUCTION

The regularizing properties of the kinetic transport equation were first established in [14] for the basic Hilbertian case. Essentially, the main result therein

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states that if $f(x, v), g(x, v) \in L^2(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $D \geq 1$ is the dimension, satisfy the stationary transport relation

$$(1.1) \quad v \cdot \nabla_x f(x, v) = g(x, v),$$

in the sense of distributions, then, for any given $\phi(v) \in C_0^\infty(\mathbb{R}^D)$, the velocity average verifies that

$$(1.2) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in H^{\frac{1}{2}}(\mathbb{R}_x^D).$$

This kind of smoothing property was then further investigated in [13], where the Hilbertian case was refined and some extensions to the non-Hilbertian case $f, g \in L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, with $p \neq 2$, were obtained, but these were not optimal. The first general results in the non-Hilbertian case establishing an optimal gain of regularity were obtained in [12] using Besov spaces and interpolation theory. The methods employed therein were not optimal in all aspects, though, but they were robust and thus allowed to further extend the averaging lemmas to settings bearing much more generality and still exhibiting an optimal gain of regularity. To be precise, the results from [12] were able to treat the case

$$(1.3) \quad v \cdot \nabla_x f(x, v) = (1 - \Delta_x)^\alpha (1 - \Delta_v)^\beta g(x, v),$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $f \in L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ and $g \in L^q(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, with $1 < p, q < \infty$. Another interesting generalization from [12] concerned the case where $f, g \in L^q(\mathbb{R}_v^D; L^p(\mathbb{R}_x^D))$ satisfy the transport equation (1.1) with $1 < q \leq p < \infty$.

The corresponding results in standard Sobolev spaces were then obtained in [7], but the methods of proof were complicated and relied on harmonic analysis on product spaces. Unfortunately, as pointed out in [28], the proofs in [7] were flawed in the non-homogeneous cases, i.e. in the case $f \in L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ and $g \in L^q(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, with $1 < p, q < \infty$, $p \neq q$, and in the case $f, g \in L^q(\mathbb{R}_v^D; L^p(\mathbb{R}_x^D))$, with $1 < q < p < \infty$.

Note that another general approach, which yielded similar results in abstract interpolation spaces resulting from the real interpolation of Besov spaces, was developed in [18].

The interesting case of velocity averaging where $f(x, v)$ and $g(x, v)$ have more local integrability in v than in x , i.e. $f, g \in L^p(\mathbb{R}_x^D; L^q(\mathbb{R}_v^D))$ with $1 < p < q < \infty$, was not addressed until [28]. To be precise, the simple question raised therein was: is it possible to improve, at least locally, the properties of the velocity averages in the case $f, g \in L^p(\mathbb{R}_x^D; L^q(\mathbb{R}_v^D))$, with $p \leq q$, with respect to the case $f, g \in L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$? The answer from [28] was definitely affirmative, even though it did not provide a general approach to this setting. Some other similar but very specific cases were treated in [18].

In the present work, we provide a very general and robust method to treat some of the cases where the local integrability in velocity is improved. Our approach has its own limitations, though, and doesn't apply to the whole range of parameters $1 \leq p \leq q \leq \infty$. However, as discussed later on, it exhibits some optimality in the range of parameters where it is valid.

The main idea in our analysis consists in utilizing the so-called dispersive properties of the kinetic transport equation, together with standard averaging methods. This type of dispersive estimates was first used in [5] and further developed

in [8, 21]. It is at first uncertain whether these dispersive properties are even related to velocity averaging lemmas. However, our work clearly establishes a rather strong link between these two properties.

Most of the above-mentioned developments came to include more cases that were imposed by the underlying applications. In particular, one of the last improvements of averaging lemmas from [15] was motivated by the hydrodynamic limit of the Boltzmann equation [16], where it was necessary to have a refined averaging lemma in $L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D)$. This was maybe the first time that some dispersive estimates were used in the proof of an averaging lemma. This paper is also somewhat motivated by the hydrodynamic limit of the Boltzmann equation but in the case of long-range interactions [1, 2]. Thus, our work allows to obtain an extension of the averaging lemma in $L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ from [15], which is crucially employed in [1, 2]. The main novelty in the L^1 averaging lemmas compared to [15] is that we are able to include derivatives in the right-hand side of the transport equation, i.e. we consider the relation (1.3) rather than (1.1). This is actually crucial when dealing with the Boltzmann equation in the presence of long-range interactions, since the Boltzmann collision operator is only defined (after renormalization) in the sense of distributions as a singular integral operator. We present this application of our main results to the theory of hydrodynamic limits in Section 5. See also [4] for a different approach to the same problem based on hypoellipticity.

2. TECHNICAL TOOLS

The results in this article rely on the theory of function spaces and interpolation. Thus, we present in this preliminary section the main tools and notations that will be utilized throughout this work and we refer the reader to [6, 27] for more details on the subjects.

2.1. Besov spaces. We will denote the Fourier transform

$$(2.1) \quad \hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^D} e^{-i\xi \cdot v} f(v) dv$$

and its inverse

$$(2.2) \quad \tilde{g}(v) = \mathcal{F}^{-1}g(v) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} e^{iv \cdot \xi} g(\xi) d\xi.$$

We introduce now a standard Littlewood-Paley decomposition of the frequency space into dyadic blocks. To this end, let $\psi(\xi), \varphi(\xi) \in C_0^\infty(\mathbb{R}^D)$ be such that

$$(2.3) \quad \begin{aligned} &\psi, \varphi \geq 0 \text{ are radial, } \quad \text{supp } \psi \subset \{|\xi| \leq 1\}, \quad \text{supp } \varphi \subset \left\{ \frac{1}{2} \leq |\xi| \leq 2 \right\} \\ &\text{and } 1 = \psi(\xi) + \sum_{k=0}^{\infty} \varphi(2^{-k}\xi), \quad \text{for all } \xi \in \mathbb{R}^D. \end{aligned}$$

Defining the scaled functions $\psi_\delta(\xi) = \psi\left(\frac{\xi}{\delta}\right)$ and $\varphi_\delta(\xi) = \varphi\left(\frac{\xi}{\delta}\right)$, for any $\delta > 0$, one has then that

$$(2.4) \quad \begin{aligned} &\text{supp } \psi_\delta \subset \{|\xi| \leq \delta\}, \quad \text{supp } \varphi_\delta \subset \left\{ \frac{\delta}{2} \leq |\xi| \leq 2\delta \right\} \\ &\text{and } 1 \equiv \psi_\delta + \sum_{k=0}^{\infty} \varphi_{\delta 2^k}. \end{aligned}$$

Furthermore, we will use the Fourier multiplier operators

$$(2.5) \quad S_\delta, \Delta_\delta : \mathcal{S}'(\mathbb{R}^D) \rightarrow \mathcal{S}'(\mathbb{R}^D)$$

(here \mathcal{S}' denotes the space of tempered distributions) defined by

$$(2.6) \quad S_\delta f = (\mathcal{F}^{-1} \psi_\delta) * f \quad \text{and} \quad \Delta_\delta f = (\mathcal{F}^{-1} \varphi_\delta) * f,$$

so that

$$(2.7) \quad S_\delta f + \sum_{k=0}^{\infty} \Delta_{\delta 2^k} f = f,$$

where the series is convergent in \mathcal{S}' .

For notational convenience, we also introduce, for every $0 < \delta_1 < \delta_2$, the operators

$$(2.8) \quad \Delta_0 = S_1 \quad \text{and} \quad \Delta_{[\delta_1, \delta_2]} = S_{\delta_2} - S_{\delta_1},$$

so that $\mathcal{F} \Delta_{[\delta_1, \delta_2]} f$ coincides with $\mathcal{F} f$ on $\{\delta_1 \leq \xi \leq \delta_2\}$ and is supported on the domain $\{\frac{\delta_1}{2} \leq \xi \leq 2\delta_2\}$.

Now, we may define the standard Besov spaces $B_{p,q}^s(\mathbb{R}^D)$, for any $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, as the subspaces of tempered distributions endowed with the norm

$$(2.9) \quad \|f\|_{B_{p,q}^s(\mathbb{R}^D)} = \left(\|\Delta_0 f\|_{L^p(\mathbb{R}^D)}^q + \sum_{k=0}^{\infty} 2^{ksq} \|\Delta_{2^k} f\|_{L^p(\mathbb{R}^D)}^q \right)^{\frac{1}{q}},$$

if $q < \infty$, and with the obvious modifications in case $q = \infty$.

We also introduce the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^D)$, for any $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, as the subspaces of tempered distributions endowed with the semi-norm

$$(2.10) \quad \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^D)} = \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \|\Delta_{2^k} f\|_{L^p(\mathbb{R}^D)}^q \right)^{\frac{1}{q}},$$

if $q < \infty$, and with the obvious modifications in case $q = \infty$.

For functions depending on two variables $x \in \mathbb{R}^D$ and $v \in \mathbb{R}^D$, we will use the Littlewood-Paley decomposition on each variable. That is, denoting $\mathbb{D} = \{0\} \cup \{2^k \in \mathbb{N} : k \in \mathbb{N}\}$, we can write

$$(2.11) \quad f(x, v) = \sum_{i,j \in \mathbb{D}} \Delta_i^x \Delta_j^v f(x, v),$$

for any $f(x, v) \in \mathcal{S}'(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where we employ the superscripts to emphasize that the multipliers Δ_i^x and Δ_j^v solely act on the respective variables x and v . Thus, we define now the mixed Besov spaces $B_{r,p,q}^{t,s}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, for any $s, t \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$, as the subspaces of tempered distributions endowed with the norm

$$(2.12) \quad \|f\|_{B_{r,p,q}^{t,s}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} = \left(\sum_{i,j \in \mathbb{D}} (1 \vee i)^{tq} (1 \vee j)^{sq} \left\| \Delta_i^x \Delta_j^v f \right\|_{L^r(\mathbb{R}_x^D; L^p(\mathbb{R}_v^D))}^q \right)^{\frac{1}{q}},$$

if $q < \infty$, and with the obvious modifications in case $q = \infty$, where the symbol $a \vee b$, for any $a, b \in \mathbb{R}$, stands for the maximum between a and b .

Next, in addition to the spaces $L^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right)$, for any $s \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$, which are defined as L^r spaces with values in the Banach spaces $B_{p,q}^s$, we further define the spaces $\tilde{L}^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right)$ as the subspaces of tempered distributions endowed with the norm

$$\|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))} = \left(\|\Delta_0^s f\|_{L^r(\mathbb{R}_x^D; L^p(\mathbb{R}_v^D))}^q + \sum_{k=0}^{\infty} 2^{ksq} \|\Delta_{2^k}^s f\|_{L^r(\mathbb{R}_x^D; L^p(\mathbb{R}_v^D))}^q \right)^{\frac{1}{q}},$$

if $q < \infty$, and with the obvious modifications in case $q = \infty$. This kind of spaces were first introduced by Chemin and Lerner in [9] with (x, v) replaced by (t, x) (time and space) and were used in many problems related to the Navier-Stokes equations (cf. [11] for instance).

One can check easily that, if $q \geq r$, then

$$(2.14) \quad L^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right) \subset \tilde{L}^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right),$$

and that, if $q \leq r$, then

$$(2.15) \quad \tilde{L}^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right) \subset L^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right).$$

Furthermore, it holds that

$$(2.16) \quad B_{r,p,1}^{0,s} \left(\mathbb{R}_x^D \times \mathbb{R}_v^D \right) \subset \tilde{L}^r \left(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D) \right) \subset B_{r,p,\infty}^{0,s} \left(\mathbb{R}_x^D \times \mathbb{R}_v^D \right),$$

for all $s \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$.

2.2. Real interpolation theory. We present now some useful elements from real interpolation theory. We will merely discuss properties that will be useful in the sequel and we refer to [6] for more details on the subject.

First of all, we briefly recall the K -method of real interpolation. To this end, we consider any couple of normed spaces A_0 and A_1 compatible in the sense that they are embedded in a common topological vector space. The sum $A_1 + A_0$ is the normed space defined by the norm

$$(2.17) \quad \|a\|_{A_1 + A_0} = \inf_{\substack{a_0 \in A_0, a_1 \in A_1 \\ a = a_0 + a_1}} \|a_0\|_{A_0} + \|a_1\|_{A_1}.$$

For any $0 < \theta < 1$, $1 \leq q \leq \infty$, we define the normed space $[A_0, A_1]_{\theta, q} \subset A_0 + A_1$ by the norm

$$(2.18) \quad \|a\|_{[A_0, A_1]_{\theta, q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

if $q < \infty$, and

$$(2.19) \quad \|a\|_{[A_0, A_1]_{\theta, \infty}} = \sup_{t > 0} t^{-\theta} K(t, a),$$

if $q = \infty$, where the K -functional is defined, for any $a \in A_0 + A_1$, $t > 0$, by

$$(2.20) \quad K(t, a) = \inf_{\substack{a_0 \in A_0, a_1 \in A_1 \\ a = a_0 + a_1}} \|a_0\|_{A_0} + t \|a_1\|_{A_1}.$$

Then, the normed space $[A_0, A_1]_{\theta, q}$ is an exact interpolation space of exponent θ . In other words, this means that, considering another couple of compatible normed spaces B_0 and B_1 , for any operator T bounded from A_0 into B_0 and from A_1 into B_1 , the operator T is bounded from $[A_0, A_1]_{\theta, q}$ into $[B_0, B_1]_{\theta, q}$ as well, with an operator norm satisfying

$$(2.21) \quad \|T\|_{[A_0, A_1]_{\theta, q} \rightarrow [B_0, B_1]_{\theta, q}} \leq \|T\|_{A_0 \rightarrow B_0}^{1-\theta} \|T\|_{A_1 \rightarrow B_1}^{\theta}.$$

It turns out that there are other equivalent methods yielding the same interpolation spaces as the K -method. In particular, we are now briefly presenting the construction of interpolation spaces known as *espaces de moyennes* (as coined by Lions and Peetre [20]), which is equivalent to the K -method of interpolation but has the advantage of being slightly more general. We refer to [6] for more details on these methods.

Thus, for any $0 < \theta < 1$ and $1 \leq q, q_0, q_1 < \infty$ such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and any compatible couple of normed spaces A_0 and A_1 , we define the following norms

$$(2.22) \quad \inf_{\substack{a_0(s) \in A_0, a_1(s) \in A_1 \\ a = a_0(s) + a_1(s)}} \left(\|s^{-\theta} a_0(s)\|_{L^{q_0}((0, \infty), \frac{ds}{s}; A_0)} + \|s^{1-\theta} a_1(s)\|_{L^{q_1}((0, \infty), \frac{ds}{s}; A_1)} \right)$$

and

$$(2.23) \quad \inf_{\substack{a_0(s) \in A_0, a_1(s) \in A_1 \\ a = a_0(s) + a_1(s)}} \left(\|s^{-\theta} a_0(s)\|_{L^{q_0}((0, \infty), \frac{ds}{s}; A_0)}^{q_0} + \|s^{1-\theta} a_1(s)\|_{L^{q_1}((0, \infty), \frac{ds}{s}; A_1)}^{q_1} \right)^{\frac{1}{q}}.$$

It is possible to show (cf. [6, Thm 3.12.1]) that both norms above are equivalent to $\|a\|_{[A_0, A_1]_{\theta, q}}$ and thus define the same interpolation space.

3. MAIN RESULTS

In this section, we present our main results.

3.1. Velocity averaging in $L_x^1 L_v^p$, inhomogeneous case. Our first result concerns the endpoint case $L_x^1 L_v^p$.

Theorem 3.1. *Let $f(x, v) \in \tilde{L}^1(\mathbb{R}_x^D; B_{p, q}^{\alpha}(\mathbb{R}_v^D))$, where $1 \leq p, q \leq \infty$ and $\alpha > -D(1 - \frac{1}{p}) > -1$, be such that*

$$(3.1) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in \tilde{L}^1(\mathbb{R}_x^D; B_{p, q}^{\beta}(\mathbb{R}_v^D))$, where $\beta \in \mathbb{R}$.

If $\beta + D(1 - \frac{1}{p}) < 1$, then

$$(3.2) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{p, q}^s(\mathbb{R}_x^D),$$

where $s = \frac{\alpha + D(1 - \frac{1}{p})}{1 + \alpha - \beta} - D(1 - \frac{1}{p})$, and the following estimate holds

$$(3.3) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{p, q}^s(dx)} \leq C \left(\|f\|_{\tilde{L}^1(dx; B_{p, q}^{\alpha}(dv))} + \|g\|_{\tilde{L}^1(dx; B_{p, q}^{\beta}(dv))} \right),$$

where the constant $C > 0$ only depends on fixed parameters.

If $\beta + D \left(1 - \frac{1}{p}\right) > 1$ or, if $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q = 1$, then

$$(3.4) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{p, \infty}^s \left(\mathbb{R}_x^D \right),$$

where $s = 1 - D \left(1 - \frac{1}{p}\right)$, and the following estimate holds

$$(3.5) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{p, \infty}^s(dx)} \leq C \left(\left\| \Delta_0^{x, v} f \right\|_{L^1(dx; L^p(dv))} + \|g\|_{\tilde{L}^1(dx; B_{p, q}^\beta(dv))} \right),$$

where the constant $C > 0$ only depends on fixed parameters.

Notice that, with the sole bound $f(x, v) \in \tilde{L}^1 \left(\mathbb{R}_x^D; B_{p, q}^\alpha \left(\mathbb{R}_v^D \right) \right)$, and in particular without any information on the transport equation (3.1), it is only possible to deduce, by Sobolev embedding, that the velocity average satisfies, for any $\phi(v) \in C_0^\infty \left(\mathbb{R}^D \right)$,

$$(3.6) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in L^1 \left(\mathbb{R}_x^D \right) \subset B_{p, \infty}^{-D \left(1 - \frac{1}{p}\right)} \left(\mathbb{R}_x^D \right).$$

Therefore, in the above theorem, the gain on the velocity average can be measured by the difference between the regularity index s in (3.3) and (3.5), and the regularity index $-D \left(1 - \frac{1}{p}\right)$ obtained by Sobolev embedding. Thus, as long as

$\beta + D \left(1 - \frac{1}{p}\right) < 1$, the above theorem yields a gain of regularity of $\frac{\alpha + D \left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}$.

Note that this gain approaches one full derivative as β tends to $1 - D \left(1 - \frac{1}{p}\right)$, or as α tends to infinity. However, since the transport operator is a differential operator of order one, it can never yield a net gain of regularity greater than one. This is precisely the reason why, when $\beta + D \left(1 - \frac{1}{p}\right) \geq 1$, the averaging lemma saturates and only yields a maximal gain of one full derivative, independently of α .

Very loosely speaking, this result shows that the transport operator $v \cdot \nabla_x$ is fully invertible when g is very regular in velocity. Thus, it becomes an elliptic operator through velocity averaging. This is also the reason why, quite remarkably, only the low frequencies of f are involved in this case.

Notice also that the condition $\alpha > -D \left(1 - \frac{1}{p}\right)$ above is very natural, since otherwise the gain $\frac{\alpha + D \left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}$ becomes negative and thus the averaging lemma turns out to be weaker than the Sobolev embedding.

However, the condition $D \left(1 - \frac{1}{p}\right) < 1$ seems less natural. Actually, its necessity comes from the handling of the low velocity frequencies of $g(x, v)$ (it can be interpreted as $\beta + D \left(1 - \frac{1}{p}\right) < 1$ with $\beta = 0$ for those low frequencies). Thus, it is possible to remove this condition by considering a corresponding version of Theorem 3.1 for homogeneous Besov spaces, which is the content of Theorem 3.3 below.

The next result extends the preceding theorem to the case including spatial derivatives in the right-hand side.

Theorem 3.2. Let $f(x, v) \in B_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $1 \leq p, q \leq \infty$, $a \in \mathbb{R}$ and $\alpha > -D \left(1 - \frac{1}{p}\right) > -1$, be such that

$$(3.7) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $\beta \in \mathbb{R}$ and $b \geq a - 1$.

If $\beta + D \left(1 - \frac{1}{p}\right) < 1$, then

$$(3.8) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{p,q}^s(\mathbb{R}_x^D),$$

where $s = (1 + b - a) \frac{\alpha + D \left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta} + a - D \left(1 - \frac{1}{p}\right)$, and the following estimate holds

$$(3.9) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{p,q}^s(dx)} \leq C \left(\|f\|_{B_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C > 0$ only depends on fixed parameters.

If $\beta + D \left(1 - \frac{1}{p}\right) > 1$ or, if $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q = 1$, then

$$(3.10) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{p,q}^s(\mathbb{R}_x^D),$$

where $s = (1 + b - a) + a - D \left(1 - \frac{1}{p}\right)$, and the following estimate holds

$$(3.11) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{p,q}^s(dx)} \leq C \left(\|\Delta_0^{x,v} f\|_{L^1(dx; L^p(dv))} + \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C > 0$ only depends on fixed parameters.

If $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q \neq 1$, then, for every $\epsilon > 0$,

$$(3.12) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{p,q}^{s-\epsilon}(\mathbb{R}_x^D),$$

where $s = (1 + b - a) + a - D \left(1 - \frac{1}{p}\right)$, and the following estimate holds

$$(3.13) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{p,q}^{s-\epsilon}(dx)} \leq C \left(\|f\|_{B_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C > 0$ only depends on fixed parameters, in particular on $\epsilon > 0$.

The remarks formulated above about Theorem 3.1 are still valid here regarding Theorem 3.2.

Thus, as long as $\beta + D \left(1 - \frac{1}{p}\right) < 1$, the above theorem yields a gain of regularity of $(1 + b - a) \frac{\alpha + D \left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}$ (compared to the Sobolev embedding). Notice that this gain approaches a derivative of order $(1 + b - a)$ as β tends to $1 - D \left(1 - \frac{1}{p}\right)$, or as α tends to infinity, which is optimal for a differential operator of order one. Therefore, the averaging lemma saturates beyond the value $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and only yields at most a gain of $1 + b - a$ derivatives.

We do not know whether it is possible to achieve a full gain of $1 + b - a$ derivatives in the case $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q \neq 1$. Nevertheless, the cases $\beta +$

$D\left(1 - \frac{1}{p}\right) > 1$ or $\beta + D\left(1 - \frac{1}{p}\right) = 1$ and $q = 1$ do yield an exact full gain of $1 + b - a$ derivatives, independently of α , which, again, is largely optimal. Quite remarkably, this is the first time that a velocity averaging result achieves exactly the maximal gain of regularity, i.e. one full derivative in the case $a = b = 0$, say. Moreover, it is worth noting that only the low frequencies of f are involved in this case, which, very loosely speaking, shows that the transport operator $v \cdot \nabla_x$ is fully invertible when g is very regular in velocity.

Notice also that the conditions $\alpha > -D\left(1 - \frac{1}{p}\right)$ and $b \geq a - 1$ above are very natural, since otherwise the gain $(1 + b - a) \frac{\alpha + D\left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}$ becomes negative and thus the averaging lemma turns out to be weaker than the Sobolev embedding.

Finally, as for Theorem 3.1, the condition $D\left(1 - \frac{1}{p}\right) < 1$ seems less natural. Actually, its necessity comes from the handling of the low velocity frequencies of $g(x, v)$ (it can be interpreted as $\beta + D\left(1 - \frac{1}{p}\right) < 1$ with $\beta = 0$ for those low frequencies). Thus, it is possible to remove this condition by considering a corresponding version of Theorem 3.2 for homogeneous Besov spaces, which is the content of Theorem 3.4 below.

3.2. Velocity averaging in $L_x^1 L_v^p$, homogeneous case. This section contains the results in the endpoint case $L_x^1 L_v^p$ formulated with homogeneous Besov spaces and corresponding to Theorems 3.1 and 3.2.

Theorem 3.3. *Let $f(x, v) \in \tilde{L}^1\left(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D)\right)$, where $1 \leq p, q \leq \infty$ and $\alpha > -D\left(1 - \frac{1}{p}\right)$, be such that*

$$(3.14) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in \tilde{L}^1\left(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D)\right)$, where $\beta < 1 - D\left(1 - \frac{1}{p}\right)$.

Then,

$$(3.15) \quad \int_{\mathbb{R}^D} f(x, v) dv \in \dot{B}_{p,q}^s(\mathbb{R}_x^D),$$

where $s = \frac{\alpha + D\left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta} - D\left(1 - \frac{1}{p}\right)$, and the following estimate holds

$$(3.16) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{\dot{B}_{p,q}^s(dx)} \leq C \|f\|_{\tilde{L}^1(dx; \dot{B}_{p,q}^\alpha(dv))}^{\frac{1 - \beta - D\left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}} \times \|g\|_{\tilde{L}^1(dx; \dot{B}_{p,q}^\beta(dv))}^{\frac{\alpha + D\left(1 - \frac{1}{p}\right)}{1 + \alpha - \beta}},$$

where the constant $C > 0$ only depends on fixed parameters.

Theorem 3.4. *Let $f(x, v) \in \dot{B}_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $1 \leq p, q \leq \infty$, $a \in \mathbb{R}$ and $\alpha > -D\left(1 - \frac{1}{p}\right)$, be such that*

$$(3.17) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in \dot{B}_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $b \geq a - 1$ and $\beta < 1 - D\left(1 - \frac{1}{p}\right)$.

Then,

$$(3.18) \quad \int_{\mathbb{R}^D} f(x, v) dv \in \dot{B}_{p,q}^s(\mathbb{R}_x^D),$$

where $s = (1 + b - a) \frac{\alpha + D(1 - \frac{1}{p})}{1 + \alpha - \beta} + a - D(1 - \frac{1}{p})$, and the following estimate holds

$$(3.19) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{\dot{B}_{p,q}^s(dx)} \leq C \|f\|_{\dot{B}_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{1-\beta-D(1-\frac{1}{p})}{1+\alpha-\beta}} \times \|g\|_{\dot{B}_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}},$$

where the constant $C > 0$ only depends on fixed parameters.

Furthermore, if $\beta + D(1 - \frac{1}{p}) = 1$ and $q = 1$, then the above estimate remains valid.

3.3. The classical $L_x^2 L_v^2$ case revisited. We give now a new very general version of the classical velocity averaging lemma in $L_x^2 L_v^2$ in terms of Besov spaces. The proofs of this formulation have the advantage of employing the same principles and ideas as in the $L_x^1 L_v^p$ cases. In particular, they are based on the same decompositions and operators (provided by Proposition 4.1 below), which will be crucial in order to carry out interpolation arguments between the $L_x^1 L_v^p$ and $L_x^2 L_v^2$ cases later on.

Note that, as usual, exploiting the Hilbertian structure of Besov spaces (i.e. choosing $q = 2$), the theorem below can readily be reformulated in terms of more standard Sobolev spaces.

Theorem 3.5. Let $f \in B_{2,2,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $a \in \mathbb{R}$, $\alpha > -\frac{1}{2}$ and $1 \leq q \leq \infty$, be such that

$$(3.20) \quad v \cdot \nabla_x f = g$$

for some $g \in B_{2,2,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $b \geq a - 1$ and $\beta \in \mathbb{R}$.

If $\beta < \frac{1}{2}$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$,

$$(3.21) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{2,q}^s(\mathbb{R}_x^D),$$

where $s = (1 + b - a) \frac{\alpha + \frac{1}{2}}{1 + \alpha - \beta} + a$, and the following estimate holds

$$(3.22) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{2,q}^s(\mathbb{R}_x^D)} \leq C_\phi \left(\|f\|_{B_{2,2,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{2,2,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters.

If $\beta > \frac{1}{2}$ or, if $\beta = \frac{1}{2}$ and $q = 1$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$,

$$(3.23) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{2,q}^s(\mathbb{R}_x^D),$$

where $s = 1 + b$, and the following estimate holds

$$(3.24) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{2,q}^s(\mathbb{R}_x^D)} \leq C_\phi \left(\|\Delta_0^{x,v} f\|_{L^2(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{2,2,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters.

If $\beta = \frac{1}{2}$ and $q \neq 1$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$ and every $\epsilon > 0$,

$$(3.25) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{2,q}^{s-\epsilon}(\mathbb{R}_x^D),$$

where $s = 1 + b$, and the following estimate holds

$$(3.26) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{2,q}^{s-\epsilon}(\mathbb{R}_x^D)} \leq C_\phi \left(\|f\|_{B_{2,2,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{2,2,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters, in particular on $\epsilon > 0$.

Notice that the theorem above provides a net gain of regularity of $(1 + b - a) \frac{\alpha + \frac{1}{2}}{1 + \alpha - \gamma}$ derivatives, where $\gamma = \min \left\{ \beta, \frac{1}{2} \right\}$. Therefore, the restrictions $\alpha > -\frac{1}{2}$ and $1 + b - a \geq 0$ on the parameters are, in fact, quite natural since the gain of regularity would possibly be negative otherwise.

Furthermore, the threshold at the value $\beta = \frac{1}{2}$ stems from the fact that, since the transport operator is a differential operator of order one, it cannot yield, in the case $a = b = 0$ say, a gain of regularity which would be superior to one full derivative.

In other words, necessarily $\frac{\alpha + \frac{1}{2}}{1 + \alpha - \gamma} \leq 1$, which implies $\gamma \leq \frac{1}{2}$.

Quite remarkably, as for Theorem 3.2, the above theorem does achieve the maximal gain of regularity of $1 + b - a$ derivatives in the cases $\beta > \frac{1}{2}$ or $\beta = \frac{1}{2}$ and $q = 1$, independently of α , which is unprecedented. Moreover, it is worth noting that only the low frequencies of f are involved in this case, which, very loosely speaking, shows that the transport operator $v \cdot \nabla_x$ is fully invertible when g is very regular in velocity.

3.4. The $L_x^1 L_v^p$ and $L_x^2 L_v^2$ cases reconciled. The following theorem results from a simple interpolation between Theorems 3.2 and 3.5. A more general, but far more complicated, interpolation procedure will yield the more general Theorem 3.7 below.

Theorem 3.6. Let $f(x, v) \in B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $1 \leq r \leq p \leq \infty$, $1 \leq q < \infty$, $a \in \mathbb{R}$ and $\alpha > \frac{1}{r} - 1 - D \left(\frac{1}{r} - \frac{1}{p} \right) > -\frac{1}{r}$, be such that

$$(3.27) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $\beta \in \mathbb{R}$ and $b \geq a - 1$.

If $\beta < \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$,

$$(3.28) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{p,q}^s(\mathbb{R}_x^D),$$

where $s = (1 + b - a) \frac{1 + \alpha - \left(\frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right) \right)}{1 + \alpha - \beta} + a - D \left(\frac{1}{r} - \frac{1}{p} \right)$, and the following estimate holds

$$(3.29) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{p,q}^s(dx)} \leq C_\phi \left(\|f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters.

If $\beta > \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ or, if $\beta = \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ and $q = 1$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$,

$$(3.30) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{p,q}^s(\mathbb{R}_x^D),$$

where $s = 1 + b - D \left(\frac{1}{r} - \frac{1}{p} \right)$, and the following estimate holds

$$(3.31) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{p,q}^s(dx)} \leq C_\phi \left(\|\Delta_0^{x,v} f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters.

If $\beta = \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ and $q \neq 1$, then, for any $\phi \in C_0^\infty(\mathbb{R}^D)$ and every $\epsilon > 0$,

$$(3.32) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{p,q}^{s-\epsilon}(\mathbb{R}_x^D),$$

where $s = 1 + b - D \left(\frac{1}{r} - \frac{1}{p} \right)$, and the following estimate holds

$$(3.33) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{B_{p,q}^{s-\epsilon}(dx)} \leq C_\phi \left(\|f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right),$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters, in particular on $\epsilon > 0$.

Notice that the theorem above corresponds exactly to Theorems 3.2 and 3.5 in the limiting cases $r = 1$ and $r = 2$, respectively. It provides a net gain of regularity, compared to the Sobolev embedding, of $(1 + b - a) \frac{1+\alpha - \left(\frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)\right)}{1+\alpha-\gamma}$ derivatives, where $\gamma = \left\{ \beta, \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right) \right\}$. Therefore, the restrictions $\alpha > \frac{1}{r} - 1 - D \left(\frac{1}{r} - \frac{1}{p} \right)$ and $1 + b - a \geq 0$ on the parameters are, in fact, quite natural since the gain of regularity would possibly be negative otherwise.

Furthermore, the threshold at the value $\beta = \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ stems from the fact that, since the transport operator is a differential operator of order one, it cannot yield, in the case $a = b = 0$ say, a gain of regularity which would be superior to one full derivative. In other words, necessarily $\frac{1+\alpha - \left(\frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)\right)}{1+\alpha-\gamma} \leq 1$, which implies $\gamma \leq \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$.

Quite remarkably, as for Theorems 3.2 and 3.5, the above theorem does achieve the maximal gain of regularity of $1 + b - a$ derivatives in the cases $\beta > \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ or $\beta = \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$ and $q = 1$, independently of α , which is unprecedented. Moreover, it is worth noting that only the low frequencies of f are involved in this case, which, very loosely speaking, shows that the transport operator $v \cdot \nabla_x$ is fully invertible when g is very regular in velocity.

The following theorem is the most general result presented in this work. However, it does not contain all the previous theorems. It follows from a general abstract interpolation procedure of the preceding results.

Theorem 3.7. Let $f(x, v) \in B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $1 \leq r_0 \leq p_0 \leq r'_0 \leq \infty$, $1 \leq q_0 < \infty$, $a \in \mathbb{R}$ and $\alpha > \frac{1}{r_0} - 1 - D \left(\frac{1}{r_0} - \frac{1}{p_0} \right)$, be such that

$$(3.34) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in B_{r_1, p_1, q_1}^{b, \beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $1 \leq r_1 \leq p_1 \leq r'_1 \leq \infty$, $1 \leq q_1 < \infty$, $b \in \mathbb{R}$ and $\beta < \frac{1}{r_1} - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)$ satisfy

$$(3.35) \quad \frac{2}{r_1} - 1 - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right) > 0 \quad \text{or} \quad p_1 = r_1 = 2$$

and

$$(3.36) \quad (1 - \theta)\frac{1}{p_0} + \theta\frac{1}{p_1} = (1 - \theta)\frac{1}{q_0} + \theta\frac{1}{q_1},$$

where

$$(3.37) \quad \theta = \frac{\left[\alpha + 1 - \frac{1}{r_0} + D\left(\frac{1}{r_0} - \frac{1}{p_0}\right)\right]}{\left[\alpha + 1 - \frac{1}{r_0} + D\left(\frac{1}{r_0} - \frac{1}{p_0}\right)\right] + \left[-\beta + \frac{1}{r_1} - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)\right]} \in (0, 1).$$

Then, for any $\chi, \phi \in C_0^\infty(\mathbb{R}^D)$,

$$(3.38) \quad \int_{\mathbb{R}^D} f(x, v) \chi(x) \phi(v) dv \in B_{p, p}^s(\mathbb{R}_x^D),$$

where

$$(3.39) \quad \begin{aligned} s &= (1 - \theta) \left(a - D\left(\frac{1}{r_0} - \frac{1}{p_0}\right) \right) + \theta \left(b - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right) \right) + \theta, \\ \frac{1}{p} &= (1 - \theta)\frac{1}{p_0} + \theta\frac{1}{p_1} = (1 - \theta)\frac{1}{q_0} + \theta\frac{1}{q_1}, \end{aligned}$$

and the following estimate holds

$$(3.40) \quad \begin{aligned} &\left\| \int_{\mathbb{R}^D} f(x, v) \chi(x) \phi(v) dv \right\|_{B_{p, p}^s(dx)} \\ &\leq C_\phi \left(\|f\|_{B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + \|g\|_{B_{r_1, p_1, q_1}^{b, \beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \right), \end{aligned}$$

where the constant $C_\phi > 0$ only depends on ϕ and other fixed parameters.

Furthermore, if $p \geq r_0$, the space localization through a cutoff $\chi(x)$ is not necessary and it is possible to take $\chi(x) \equiv 1$ in the above statements.

The interpretation of net gain of regularity is not as straightforward as it is for the preceding theorems. Thus, we provide now a somewhat alternative analysis of the regularity index s .

The above theorem essentially establishes an estimate on the velocity average which stems from an interpolation of order θ between the controls

$$(3.41) \quad \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \in B_{r_0, q_0}^a(\mathbb{R}_x^D) \subset B_{p_0, q_0}^{a-D\left(\frac{1}{r_0} - \frac{1}{p_0}\right)}(\mathbb{R}_x^D)$$

and

$$(3.42) \quad \int_{\mathbb{R}^D} g(x, v) \phi(v) dv \in B_{r_1, q_1}^b(\mathbb{R}_x^D) \subset B_{p_1, q_1}^{b-D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)}(\mathbb{R}_x^D),$$

which follow from standard Sobolev embeddings. If the functions $f(x, v)$ and $g(x, v)$ were linked by a relation $f = Tg$, where T is some bounded and invertible operator of differential order $r \in \mathbb{R}$ acting only on x , then it would be natural to

expect, by interpolation of order θ , a control on the velocity average in the Besov space

$$(3.43) \quad B_{p,p}^{(1-\theta)\left(a-D\left(\frac{1}{r_0}-\frac{1}{p_0}\right)\right)+\theta\left(b-D\left(\frac{1}{r_1}-\frac{1}{p_1}\right)+r\right)}\left(\mathbb{R}_x^D\right),$$

where $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}$. This formal reasoning shows that the net gain of regularity given by a differential relation $f = Tg$ of order r through an interpolation of order θ is at most θr , when compared to a differential operator of zero (e.g. the identity).

From that viewpoint, the above theorem asserts that, the transport operator $T = v \cdot \nabla_x$ being a differential operator of order one, it is possible to obtain a maximal net gain of regularity θ through velocity averaging. We insist that here the net gain of regularity is found by comparing the actual regularity index s with the interpolation of the indices obtained by Sobolev embeddings.

Therefore, the restriction $\alpha > \frac{1}{r_0} - 1 - D\left(\frac{1}{r_0} - \frac{1}{p_0}\right)$ on the parameters is, in fact, quite natural since the gain of regularity θ would possibly be negative otherwise.

Furthermore, the constraint $\beta < \frac{1}{r_1} - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)$ stems from the fact that, since the transport operator is a differential operator of order one, it cannot yield a gain of regularity θ which would be superior to one full derivative. In other words, necessarily $\theta < 1$, which implies $\beta < \frac{1}{r_1} - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)$.

It is quite interesting to note that some kind of space localization is definitely necessary in the case $p < r_0$ of the above theorem. It is in fact explicit from the proofs that this restriction comes from the control of low frequencies. Indeed, let us suppose that the estimate (3.40) holds with $\chi \equiv 1$ for some given choice of parameters in the one dimensional case $D = 1$. Further consider $f(x, v) \in \mathcal{S}(\mathbb{R}_x \times \mathbb{R}_v)$ such that its space and velocity frequencies are localized in a bounded domain. In other words, we suppose that $\Delta_{2^k}^x f = \Delta_{2^k}^v f = 0$, for every $k \geq 0$, say. Therefore, in virtue of estimate (3.40), it holds that

$$(3.44) \quad \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \leq C_\phi \left(\|f\|_{L_x^{r_0} L_v^{p_0}} + \|v \partial_x f\|_{L_x^{r_1} L_v^{p_1}} \right).$$

In particular, since the transformation $f_R(x, v) = f\left(\frac{x}{R}, v\right)$ preserves de localization of low frequencies for any $R > 1$, we deduce that it must also hold that

$$(3.45) \quad \begin{aligned} R^{\frac{1}{p}} \left\| \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} &= \left\| \int_{\mathbb{R}^D} f_R(x, v) \phi(v) dv \right\|_{L_x^p} \\ &\leq C_\phi \left(\|f_R\|_{L_x^{r_0} L_v^{p_0}} + \|v \partial_x f_R\|_{L_x^{r_1} L_v^{p_1}} \right) \\ &= C_\phi \left(R^{\frac{1}{r_0}} \|f\|_{L_x^{r_0} L_v^{p_0}} + R^{\frac{1}{r_1}-1} \|v \partial_x f\|_{L_x^{r_1} L_v^{p_1}} \right). \end{aligned}$$

It follows that $R^{\frac{1}{p}-\frac{1}{r_0}}$ must remain bounded as R tends towards infinity, which forces $r_0 \leq p$.

Finally, we would like to emphasize that we have chosen to present, in this work, cases of velocity averaging lemmas dealing with $L_x^r L_v^p$ integrability only, where $r \leq p$, principally because our analysis of dispersion allowed to handle

these previously unsettled cases. But we insist that this is by no means a restriction of our method, which is, in fact, very robust and enables to also treat the actually easier setting of $L_v^p L_x^r$ integrability, where $p \leq r$, and thus, to recover most of previously known results in sharper Besov spaces. Our precise interpolation techniques can even reach settings where the left-hand side f enjoys $L_x^{r_0} L_v^{p_0}$ integrability, where $r_0 \leq p_0$, while the right-hand side g displays $L_v^{p_1} L_x^{r_1}$ integrability, where $p_1 \leq r_1$, and vice versa.

4. ELLIPTICITY, DISPERSION AND AVERAGING

Here, we explain the concepts which will lead to the proofs of the main results in this work.

The classical theory of velocity averaging lemmas in $L_{x,v}^2$, first developed in [13, 14], is based on a simple but ingenious microlocal decomposition. More precisely, if $f(x, v), g(x, v) \in L^2(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ satisfy the transport relation (1.1) then, considering the Fourier transforms $\hat{f}(\eta, v)$ and $\hat{g}(\eta, v)$ in the space variable only, it holds that

$$(4.1) \quad iv \cdot \eta \hat{f}(\eta, v) = \hat{g}(\eta, v).$$

Therefore, it is possible to exploit some ellipticity of the transport operator as long as one remains on an appropriate microlocal domain. In other words, we may invert the transport operator as long as the quantity $|v \cdot \eta|$ remains uniformly bounded away from zero:

$$(4.2) \quad \hat{f}(\eta, v) = \frac{1}{iv \cdot \eta} \hat{g}(\eta, v), \quad \text{on } \{|v \cdot \eta| > 1\}, \text{ say.}$$

Thus, introducing some cutoff function $\rho \in \mathcal{S}(\mathbb{R})$ (\mathcal{S} denotes the Schwartz space of rapidly decaying functions) such that $\rho(0) = 1$ and an interpolation parameter $t > 0$, we may decompose

$$(4.3) \quad \hat{f}(\eta, v) = \rho(tv \cdot \eta) \hat{f}(\eta, v) + \frac{1 - \rho(tv \cdot \eta)}{iv \cdot \eta} \hat{g}(\eta, v).$$

It follows that each term in the right-hand side may then be estimated locally in L_v^2 and the remainder of the proof simply consists in choosing the optimal value for the interpolation parameter t (which will depend on η). The conclusion of this method yields that, for every test function $\phi(v) \in C_0^\infty$, the velocity average $\int f(x, v) \phi(v) dv$ belongs to $H_x^{\frac{1}{2}}$. This approach yields optimal results and exhibits the crucial regularizing properties of the transport operator, which, we insist, is based on exploiting some partial ellipticity.

As mentioned before, several extensions of this method are possible (cf. [7, 12]), in particular, the $L_{x,v}^p$ case of velocity averaging lemmas is obtained by interpolating the preceding $L_{x,v}^2$ result with the degenerate case in $L_{x,v}^1$. Indeed, if $f(x, v), g(x, v) \in L_{x,v}^1$, then absolutely no regularity may be gained on the velocity averages from the transport equation, which is unfortunately optimal as far as the gain of regularity is concerned.

In this work, we obtain refined velocity averaging results by further exploiting the dispersive properties of the transport operator discovered by Castella and

Perthame in [8]. This requires the development of a suitable interpolation formula, more refined than (4.3), and the study of its properties. Thus, introducing an interpolation parameter $t > 0$, it trivially holds, from (1.1), that

$$(4.4) \quad \begin{cases} (\partial_t + v \cdot \nabla_x) f = g, \\ f(t=0) = f. \end{cases}$$

Hence the interpolation formula,

$$(4.5) \quad f(x, v) = f(x - tv, v) + \int_0^t g(x - sv, v) ds,$$

which is in fact dual to the interpolation formula employed in [15] and is merely Duhamel's representation formula for the time dependent transport equation.

Furthermore, considering any $f \in \mathcal{S}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, $\chi \in \mathcal{S}(\mathbb{R}^D)$ and denoting $\chi_\lambda(\cdot) = \frac{1}{\lambda^D} \chi(\frac{\cdot}{\lambda})$, where $\lambda > 0$, one easily verifies the following rule of action of convolutions on velocity averages,

$$(4.6) \quad \begin{aligned} \chi_\lambda *_{x} \int_{\mathbb{R}^D} f(x - tv, v) dv &= \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{1}{\lambda^D} \chi\left(\frac{x-y}{\lambda}\right) f(y - tv, v) dv dy \\ &= \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{1}{\left(\frac{\lambda}{t}\right)^D} \chi\left(\frac{w-v}{\frac{\lambda}{t}}\right) f(x - tw, v) dw dv \\ &= \int_{\mathbb{R}^D} \left(\chi_{\frac{\lambda}{t}} *_{v} f\right)(x - tw, w) dw, \end{aligned}$$

where we used the change of variables $(v, y) \mapsto (v, w = \frac{x-y}{t} + v)$.

In the notation of Section 2.1 and in particular (2.6) and (2.8), one then checks employing the above rule of action of convolutions that the dyadic frequency blocks act on velocity averages according to the identities, where $\delta > 0$,

$$(4.7) \quad \begin{aligned} \Delta_0^x \int f(x - tv, v) dv &= \int (S_t^v f)(x - tv, v) dv, \\ \Delta_\delta^x \int f(x - tv, v) dv &= \int (\Delta_{t\delta}^v f)(x - tv, v) dv. \end{aligned}$$

Next, we apply this identity to the velocity averages of the above interpolation formula (4.5) to deduce

$$(4.8) \quad \begin{aligned} \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv &= \int_{\mathbb{R}^D} (S_t^v f)(x - tv, v) dv \\ &\quad + \int_0^t \int_{\mathbb{R}^D} (S_s^v g)(x - sv, v) dv ds, \\ \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv &= \int_{\mathbb{R}^D} (\Delta_{t2^k}^v f)(x - tv, v) dv \\ &\quad + \int_0^t \int_{\mathbb{R}^D} (\Delta_{s2^k}^v g)(x - sv, v) dv ds. \end{aligned}$$

The above representation formula can be used explicitly to establish the simplest forms of our main results. It is the first key idea in our approach. Indeed, it shows that the space frequencies of the averages of $f(x, v)$ may be controlled with the velocity frequencies of $f(x, v)$ and $g(x, v)$. This property of transfer of

frequencies is linked to the hypoellipticity of the transport operator and we refer to [4] for recent developments on this matter.

Moreover, it can be used in some case to give a sense to the velocity average $\int_{\mathbb{R}^D} f(x, v) dv$ even when $f(x, v)$ is a priori not globally integrable in velocity. Indeed, supposing that we can show (using dispersive estimates for instance) that $(\Delta_{t2^k}^v f)(x - tv, v)$ and $\int_0^t (\Delta_{s2^k}^v g)(x - sv, v) ds$ actually are globally integrable in velocity for each given $t > 0$, then we may nevertheless define the velocity average $\Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv$ by the identities (4.8), and similarly for the low frequencies component. This principle is used implicitly in the statements of Theorems 3.1, 3.2, 3.3 and 3.4, but we will never render this argument explicit for the sake of simplicity.

A very simple but useful refinement of the above formulas (4.8) follows from the fact that $S_2^x \Delta_0^x = \Delta_0^x$ and $\Delta_{[2^{k-1}, 2^{k+1}]}^x \Delta_{2^k}^x = \Delta_{2^k}^x$. Thus, we obtain

$$\begin{aligned}
 \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv &= \int_{\mathbb{R}^D} (S_2^x S_t^v f)(x - tv, v) dv \\
 &\quad + \int_0^t \int_{\mathbb{R}^D} (S_2^x S_s^v g)(x - sv, v) dv ds, \\
 \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv &= \int_{\mathbb{R}^D} \left(\Delta_{[2^{k-1}, 2^{k+1}]}^x \Delta_{t2^k}^v f \right)(x - tv, v) dv \\
 &\quad + \int_0^t \int_{\mathbb{R}^D} \left(\Delta_{[2^{k-1}, 2^{k+1}]}^x \Delta_{s2^k}^v g \right)(x - sv, v) dv ds,
 \end{aligned}
 \tag{4.9}$$

which considerably decreases the set of frequencies of $f(x, v)$ and $g(x, v)$ required to control the frequencies of the velocity averages.

The formulas (4.8) and (4.9) will be used to treat the velocity averages in the $L_x^1 L_v^p$ and the $L_v^1 L_x^p$ settings, which are endpoint cases. As usual, the more general cases will then be obtained by interpolation with the classical $L_{x,v}^2$ case of velocity averaging. However, a significant obstruction to this interpolating strategy lies in that the representation formulas (4.3), which exploits the elliptic properties of the transport operator, and (4.5), which is based on the dispersion of the transport operator, are of different nature and thus seem at first to be incompatible. There is however an elementary but crucial link between them which we establish now. Similar ideas are used in [4].

To this end, we first notice, recalling the Fourier inversion formula $\rho(r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{irs} \hat{\rho}(s) ds$, that it trivially holds, for any $t \in \mathbb{R}$, that

$$\hat{f}(\eta, v) \rho(t\eta \cdot v) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\eta, v) e^{ist\eta \cdot v} \hat{\rho}(s) ds,
 \tag{4.10}$$

and similarly, since $1 = \rho(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(s) ds$ and further noticing

$$- \int_0^{st} e^{i\eta \cdot v \sigma} d\sigma = \frac{1 - e^{ist\eta \cdot v}}{i\eta \cdot v},
 \tag{4.11}$$

that

$$\begin{aligned}
 \hat{g}(\eta, v) \frac{1 - \rho(t\eta \cdot v)}{i\eta \cdot v} &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\eta, v) \frac{1 - e^{ist\eta \cdot v}}{i\eta \cdot v} \hat{\rho}(s) ds \\
 &= - \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{st} \hat{g}(\eta, v) e^{i\eta \cdot v \sigma} d\sigma \hat{\rho}(s) ds.
 \end{aligned}
 \tag{4.12}$$

Then, simply taking the inverse Fourier transform of the above identities, we obtain

$$(4.13) \quad \begin{aligned} \mathcal{F}_x^{-1} \rho(t\eta \cdot v) \mathcal{F}_x f(x, v) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}_x^{-1} e^{ist\eta \cdot v} \mathcal{F}_x f(x, v) \hat{\rho}(s) ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f(x + stv, v) \hat{\rho}(s) ds, \end{aligned}$$

and

$$(4.14) \quad \begin{aligned} \mathcal{F}_x^{-1} \frac{1 - \rho(t\eta \cdot v)}{it\eta \cdot v} \mathcal{F}_x g(x, v) &= -\frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{st} \mathcal{F}_x^{-1} e^{i\sigma\eta \cdot v} \mathcal{F}_x g(x, v) d\sigma \hat{\rho}(s) ds \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{st} g(x + \sigma v, v) d\sigma \hat{\rho}(s) ds. \end{aligned}$$

Therefore, incorporating identities (4.13) and (4.14) into the interpolation formulas (4.3) or (4.5), we obtain, for each $t \in \mathbb{R}$, the following refined decomposition

$$(4.15) \quad f(x, v) = T_A^t f(x, v) + t T_B^t g(x, v),$$

where

$$(4.16) \quad \begin{aligned} T_A^t f(x, v) &= \frac{1}{2\pi} \int_{\mathbb{R}} f(x + stv, v) \hat{\rho}(s) ds \\ &= \int_{\mathbb{R}} f(x - stv, v) \tilde{\rho}(s) ds, \\ T_B^t g(x, v) &= -\frac{1}{t} \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{st} g(x + \sigma v, v) d\sigma \hat{\rho}(s) ds \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} \int_0^s g(x + \sigma tv, v) d\sigma \hat{\rho}(s) ds \\ &= \int_{\mathbb{R}} \int_0^s g(x - \sigma tv, v) d\sigma \tilde{\rho}(s) ds, \end{aligned}$$

and

$$(4.17) \quad \begin{aligned} \mathcal{F}_x T_A^t f(\eta, v) &= \rho(t\eta \cdot v) \mathcal{F}_x f(\eta, v), \\ \mathcal{F}_x T_B^t g(\eta, v) &= \frac{1 - \rho(t\eta \cdot v)}{it\eta \cdot v} \mathcal{F}_x g(\eta, v), \end{aligned}$$

which shows that formulas (4.3) and (4.5) are in fact equivalent. Indeed, it is readily seen that (4.3) follows from (4.15) by use of the Fourier transform, while (4.5) is deduced from (4.15) by setting $\tilde{\rho}(s)$ equal to an arbitrary approximation of the Dirac mass at $s = 1$, which makes sense since we have merely imposed on the cutoff that $\rho(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(s) ds = \int_{\mathbb{R}} \tilde{\rho}(s) ds = 1$. Equivalently, formula (4.15) can be derived from (4.5) by replacing t by st and then integrating against $\tilde{\rho}(s) ds$.

Notice also that, by setting $\tilde{\rho}(s) = e^{-s} \mathbb{1}_{\{s \geq 0\}}$ in (4.15), one arrives at the standard representation formula, for any $t > 0$,

$$(4.18) \quad \begin{aligned} f(x, v) &= \int_0^\infty f(x - stv, v) e^{-s} ds + t \int_0^\infty g(x - \sigma tv, v) d\sigma e^{-s} ds \\ &= \int_0^\infty \left[\frac{1}{t} f(x - sv, v) + g(x - sv, v) \right] e^{-\frac{s}{t}} ds, \end{aligned}$$

which is usually obtained by directly solving the equivalent transport equation

$$(4.19) \quad \left[\frac{1}{t} + v \cdot \nabla_x \right] f = \frac{1}{t} f + g,$$

but we will not make any use of this decomposition (cf. [15, 18] for uses of this formula). Essentially, this particular choice of cutoff is not appropriate because its frequencies are not well localized, even though its Fourier transform decays exponentially. The importance of having a strong frequencies cutoff is made fully explicit in the localization identities (4.25) of Proposition 4.1 below.

Much more importantly, this refined decomposition (4.15) shows that the elliptic and dispersive properties of the transport operator may be exploited through the same interpolation formula. Thus, employing formula (4.7) on the transfer of frequencies for velocity averages with the above decomposition, we obtain the following crucial proposition, which will systematically be the starting point of the proofs for the averaging lemmas presented in this work.

Proposition 4.1. *Let $f(x, v), g(x, v) \in \mathcal{S}(\mathbb{R}^D \times \mathbb{R}^D)$ be such that*

$$(4.20) \quad v \cdot \nabla_x f = g.$$

For all $t > 0$, $\delta \geq 0$ and for every cutoff function $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(s) ds = \int_{\mathbb{R}} \tilde{\rho}(s) ds = 1$, we consider the decomposition

$$(4.21) \quad \Delta_\delta^x \int_{\mathbb{R}^D} f(x, v) dv = A_\delta^t f(x) + t B_\delta^t g(x),$$

where the operators A_δ^t and B_δ^t are defined by

$$(4.22) \quad \begin{aligned} A_\delta^t f(x) &= \Delta_\delta^x \int_{\mathbb{R}^D} T_A^t f(x, v) dv, \\ B_\delta^t g(x) &= \Delta_\delta^x \int_{\mathbb{R}^D} T_B^t g(x, v) dv. \end{aligned}$$

Then it holds that

$$(4.23) \quad \begin{aligned} A_\delta^t f(x) &= \int_{\mathbb{R}^D} \mathcal{F}_x^{-1} \rho(t\eta \cdot v) \mathcal{F}_x \Delta_\delta^x f(x, v) dv, \\ B_\delta^t g(x) &= \int_{\mathbb{R}^D} \mathcal{F}_x^{-1} \tau(t\eta \cdot v) \mathcal{F}_x \Delta_\delta^x g(x, v) dv, \end{aligned}$$

where $\tau(s) = \frac{1-\rho(s)}{is}$ is smooth, and

$$(4.24) \quad \begin{aligned} A_\delta^t f(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} \Delta_\delta^x f(x - stv, v) dv \right] \tilde{\rho}(s) ds, \\ B_\delta^t g(x) &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} \Delta_\delta^x g(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds. \end{aligned}$$

Furthermore, if the cutoff $\rho \in \mathcal{S}(\mathbb{R})$ is such that $\tilde{\rho}$ is compactly supported inside $[1, 2]$, then, for every $\delta > 0$,

$$(4.25) \quad \begin{aligned} A_0^t f &= A_0^t (S_2^x S_{4t}^v f), & B_0^t g &= B_0^t (S_2^x S_{4t}^v g), \\ A_\delta^t f &= A_\delta^t \left(\Delta_{[\frac{\delta}{2}, 2\delta]}^x \Delta_{[\frac{t\delta}{2}, 4t\delta]}^v f \right), & B_\delta^t g &= B_\delta^t \left(\Delta_{[\frac{\delta}{2}, 2\delta]}^x S_{8t\delta}^v g \right). \end{aligned}$$

Proof. It is readily seen that identities (4.23) and (4.24) are a mere transcription in terms of velocity averages of properties (4.16) and (4.17) for the operators T_A^t and T_B^t . Therefore, we only have to justify the crucial property (4.25). To this end, notice first that property (4.7) on the transfer of spatial to velocity frequencies implies, according to (4.24), that, when $\delta > 0$,

$$\begin{aligned}
 A_0^t f(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} (S_{st}^v f)(x - stv, v) dv \right] \tilde{\rho}(s) ds, \\
 B_0^t g(x) &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (S_{\sigma st}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds, \\
 A_\delta^t f(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} (\Delta_{st\delta}^v f)(x - stv, v) dv \right] \tilde{\rho}(s) ds, \\
 B_\delta^t g(x) &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (\Delta_{\sigma st\delta}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds.
 \end{aligned}
 \tag{4.26}$$

Then, since $t \leq st \leq 2t$ and $\sigma st \leq 2t$ on the support of $\tilde{\rho}(s)$, it holds that

$$\begin{aligned}
 S_{st}^v &= S_{st}^v S_{4t}^v, & S_{\sigma st}^v &= S_{\sigma st}^v S_{4t}^v, \\
 \Delta_{st\delta}^v &= \Delta_{st\delta}^v \Delta_{[\frac{t\delta}{2}, 4t\delta]}^v, & \Delta_{\sigma st\delta}^v &= \Delta_{\sigma st\delta}^v S_{8t\delta}^v.
 \end{aligned}
 \tag{4.27}$$

Hence, employing identities (4.7) again, we find that

$$\begin{aligned}
 A_0^t f(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} (S_{st}^v S_{4t}^v f)(x - stv, v) dv \right] \tilde{\rho}(s) ds \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} (\Delta_0^x S_{4t}^v f)(x - stv, v) dv \right] \tilde{\rho}(s) ds = A_0^t (S_{4t}^v f)(x), \\
 B_0^t g(x) &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (S_{\sigma st}^v S_{4t}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds \\
 &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (\Delta_0^x S_{4t}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds = B_0^t (S_{4t}^v g)(x), \\
 A_\delta^t f(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} \left(\Delta_{st\delta}^v \Delta_{[\frac{t\delta}{2}, 4t\delta]}^v f \right) (x - stv, v) dv \right] \tilde{\rho}(s) ds \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^D} \left(\Delta_\delta^x \Delta_{[\frac{t\delta}{2}, 4t\delta]}^v f \right) (x - stv, v) dv \right] \tilde{\rho}(s) ds = A_\delta^t \left(\Delta_{[\frac{t\delta}{2}, 4t\delta]}^v f \right)(x), \\
 B_\delta^t g(x) &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (\Delta_{\sigma st\delta}^v S_{8t\delta}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds \\
 &= \int_{\mathbb{R}} \left[\int_0^1 \int_{\mathbb{R}^D} (\Delta_\delta^x S_{8t\delta}^v g)(x - \sigma stv, v) dv d\sigma \right] s \tilde{\rho}(s) ds = B_\delta^t (S_{8t\delta}^v g)(x).
 \end{aligned}
 \tag{4.28}$$

Finally, utilizing that $S_2^x \Delta_0^x = \Delta_0^x$ and $\Delta_{[\frac{t\delta}{2}, 2\delta]}^x \Delta_\delta^x = \Delta_\delta^x$ concludes the proof of the proposition. \square

5. CONTROL OF CONCENTRATIONS IN L^1

The methods developed in Section 4, which eventually led to the main results from Section 3, were initially motivated by an important application to the viscous incompressible hydrodynamic limit of the Boltzmann equation with long-range interactions in [1, 2]. We present in this section some technical lemmas which

were crucially employed therein and are contained in our main results. Essentially, we show here how the methods from Section 4 can be used to gain control over the concentrations in kinetic transport equations through velocity averaging. This application relies on a refined regularity estimate for the Boltzmann equation without cutoff established in [3], wherein the reader will also find a complete discussion of this application to the hydrodynamic limit. An alternative method has been developed in [4].

The difficult problem of concentrations in the viscous incompressible hydrodynamic limit of the Boltzmann equation was first successfully handled in [16] for bounded collision kernels with cutoff and then, using the same approach, extended in [17, 19] to more general cross-sections with cutoff. In any case, the method consisted in an application of a simple, but subtle, velocity averaging lemma in L^1 established in [15], which we briefly present now. Recall first the following definition from [15].

Definition. A bounded sequence $\{f_n(x, v)\}_{n=0}^\infty \subset L^1_{\text{loc}}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ is said to be **equi-integrable in v** if and only if for every $\eta > 0$ and each compact $K \subset \mathbb{R}^D \times \mathbb{R}^D$, there exists $\delta > 0$ such that for any measurable set $\Omega \subset \mathbb{R}^D \times \mathbb{R}^D$ with $\sup_{x \in \mathbb{R}^D} \int \mathbb{1}_\Omega(x, v) dv < \delta$, we have that

$$(5.1) \quad \int_{\Omega \cap K} |f_n(x, v)| dx dv < \eta \quad \text{for every } n.$$

The crucial compactness lemma in L^1 obtained in [15] and relevant to the rigorous derivations of the hydrodynamic limit in the cutoff case [16, 17, 19] is contained in the following theorem.

Theorem 5.1 ([15]). *Let the sequence $\{f_n(x, v)\}_{n=0}^\infty$ be bounded in $L^1_{\text{loc}}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, equi-integrable in v , and such that*

$$(5.2) \quad v \cdot \nabla_x f_n = g_n$$

for some sequence $\{g_n\}_{n=0}^\infty$ bounded in $L^1_{\text{loc}}(\mathbb{R}_x^D \times \mathbb{R}_v^D)$.

Then,

- (1) $\{f_n\}_{n=0}^\infty$ *is equi-integrable (in all variables x and v) and,*
- (2) *for each $\psi \in L^\infty(\mathbb{R}^D)$ with compact support, the family of velocity averages*

$$(5.3) \quad \int_{\mathbb{R}^D} f_n(x, v) \psi(v) dv \quad \text{is relatively compact in } L^1_{\text{loc}}(\mathbb{R}^D).$$

Unfortunately, the above result is not directly applicable to the viscous incompressible hydrodynamic limit of the Boltzmann equation with long-range interactions. Indeed, its use is prevented by the particular structure of the Boltzmann collision operator, which behaves as a nonlinear differential operator for collision kernels without cutoff (cf. [1, 2, 3]). Consequently, the strategy from [16, 17, 19] would require a generalization of the preceding theorem to kinetic transport equations (5.2) with velocity derivatives in its right-hand side.

However, a simple counterexample shows that a straight generalization won't be possible. Indeed, consider the locally integrable functions

$$(5.4) \quad \begin{aligned} f_n(x, v) &= n\varphi(nx_1) \cos(nv_1), \\ g_n(x, v) &= v_1 n\varphi'(nx_1) \sin(nv_1) \\ \text{and } h_n(x, v) &= n\varphi'(nx_1) \sin(nv_1), \end{aligned}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ has non-trivial mass. Then, one easily checks that

$$(5.5) \quad \begin{aligned} v \cdot \nabla_x f_n &= \partial_{v_1} g_n - h_n \\ \text{and } \{f_n\}_{n=0}^\infty &\text{ is equi-integrable in } v. \end{aligned}$$

However, $\{f_n\}_{n=0}^\infty$ is not equi-integrable in all variables, which shows that the first assertion of Theorem 5.1 doesn't hold if one has derivatives on the right-hand side of the transport equation (5.2).

Nevertheless, note that the velocity averages of f_n do converge strongly to zero and thus, it is unclear whether there holds a direct extension of the second assertion of Theorem 5.1 or not. Furthermore, this counterexample uses crucially the fact that f_n is allowed to alternate sign and so, it might still be possible that preventing this oscillatory behavior by simply imposing a nonnegativity condition on the f_n 's would allow for a generalization of the result.

We wish now to extend the above Theorem 5.1 in order to allow the use of derivatives in the right-hand side of the transport equation, which, again, is necessary for the rigorous derivation of the hydrodynamic limit in the non-cutoff case. It turns out that a slightly better control on the high velocity frequencies of the f_n 's will suffice to provide a relevant generalization. This is the content of the coming theorem.

Theorem 5.2. *Let $\{f_n(x, v)\}_{n=0}^\infty$ be a bounded sequence in $L^1(\mathbb{R}_x^D; B_{1,1}^0(\mathbb{R}_v^D))$, such that $f_n \geq 0$ and*

$$(5.6) \quad v \cdot \nabla_x f_n = (1 - \Delta_v)^{\frac{\gamma}{2}} g_n$$

for some sequence $\{g_n(x, v)\}_{n=0}^\infty$ bounded in $L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ and some $\gamma \in \mathbb{R}$.

Then,

$$(5.7) \quad \{f_n\}_{n=0}^\infty \text{ is equi-integrable (in all variables } x \text{ and } v).$$

The above theorem will be obtained as a corollary of the following proposition, which is, in fact, a particular case of Theorem 3.1. Nevertheless, we do provide below a complete justification for this result, because its proof is simpler and contains some of the essential ideas presented in Section 4. Thus, we hope that it will also serve as a primer to the more convoluted demonstrations of Section 6.

Proposition 5.3. *Let $f(x, v) \in L^1(\mathbb{R}_x^D; B_{1,1}^0(\mathbb{R}_v^D))$ be such that*

$$(5.8) \quad v \cdot \nabla_x f = g$$

for some $g(x, v) \in L^1(\mathbb{R}_x^D; B_{1,1}^\beta(\mathbb{R}_v^D))$, where $\beta \in \mathbb{R}$.

Then,

$$(5.9) \quad \int_{\mathbb{R}^D} f(x, v) dv \in B_{1,1}^0(\mathbb{R}_x^D),$$

and the following estimate holds

$$(5.10) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{1,1}^0(dx)} \leq C \left(\|f\|_{L^1(dx; B_{1,1}^0(dv))} + \|g\|_{L^1(dx; B_{1,1}^\beta(dv))} \right),$$

where the constant $C > 0$ only depends on fixed parameters.

Proof. Notice first that, in virtue of the inclusions $\tilde{L}^1 B_{1,1}^{\beta_1} \subset \tilde{L}^1 B_{1,1}^{\beta_2}$ whenever $\beta_1 \geq \beta_2$, we may assume without any loss of generality that $\beta < 1$. Furthermore, it is to be emphasized that $\tilde{L}^1 B_{1,1}^\beta = L^1 B_{1,1}^\beta$, for any $\beta \in \mathbb{R}$.

Integrating the dyadic interpolation formula (4.8) in space yields

$$(5.11) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^1} \leq \left\| \Delta_{t_k}^v f \right\|_{L_{x,v}^1} + \int_0^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds.$$

For each value of k , we fix now the interpolation parameter t as $t_k = 2^{k \frac{\beta}{1-\beta}}$. Thus, summing the above estimate over k yields

$$(5.12) \quad \sum_{k=0}^{\infty} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^1} \leq \sum_{k=0}^{\infty} \left\| \Delta_{t_k}^v f \right\|_{L_{x,v}^1} + \sum_{k=0}^{\infty} \int_0^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds.$$

The last term above is then handled through the following calculation, noticing that $\frac{t_{k-1}}{2} < t_k$,

$$(5.13) \quad \begin{aligned} \sum_{k=0}^{\infty} \int_0^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds &= \sum_{k=0}^{\infty} \int_0^{\frac{t_{k-1}}{2}} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds + \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds \\ &= \sum_{k=0}^{\infty} \frac{1}{2} \int_0^{t_{k-1}} \left\| \Delta_{s2^{k-1}}^v g \right\|_{L_{x,v}^1} ds + \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds \\ &= \frac{1}{2} \sum_{k=-1}^{\infty} \int_0^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds + \sum_{k=0}^{\infty} \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds, \end{aligned}$$

from which we deduce

$$(5.14) \quad \sum_{k=0}^{\infty} \int_0^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds = \int_0^{t_{(-1)}} \left\| \Delta_{\frac{s}{2}}^v g \right\|_{L_{x,v}^1} ds + 2 \sum_{k=0}^{\infty} \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds.$$

Consequently, combining (5.12) with (5.14), we infer

$$(5.15) \quad \begin{aligned} &\sum_{k=0}^{\infty} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^1} \\ &\leq \sum_{k=0}^{\infty} \left\| \Delta_{t_k}^v f \right\|_{L_{x,v}^1} + 2 \sum_{k=0}^{\infty} \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s2^k}^v g \right\|_{L_{x,v}^1} ds + C \|g\|_{\tilde{L}^1(dx; B_{1,1}^\beta(dv))}, \end{aligned}$$

where $C > 0$ is a fixed constant that only depends on fixed parameters.

Now, let us recall a basic principle from the theory of Littlewood and Paley. Consider any $\delta > 0$. Then, there exists a unique integer l_δ such that $2^{l_\delta} \leq \delta < 2^{l_\delta+1}$. In the notation of section 2.1 and specially (2.6), it then holds that $\sum_{j=-1}^2 \varphi_{2^{l_\delta+j}} \equiv 1$ on the support of φ_δ , which therefore implies for any $f \in \mathcal{S}'(\mathbb{R}^D)$ and any $1 \leq p \leq \infty$ that

$$(5.16) \quad \|\Delta_\delta f\|_{L^p} = \left\| \sum_{j=-1}^2 \Delta_{2^{l_\delta+j}} \Delta_\delta f \right\|_{L^p} \leq C \sum_{j=-1}^2 \|\Delta_{2^{l_\delta+j}} f\|_{L^p},$$

for some fixed constant $C > 0$ independent of δ .

We wish now to apply this principle to our situation. Thus, denoting by $[\cdot]$ the integer part of a number and noticing that $2^{\lfloor k \frac{1}{1-\beta} \rfloor} \leq t_k 2^k = 2^{k \frac{1}{1-\beta}} < 2^{\lfloor k \frac{1}{1-\beta} \rfloor + 1}$, we

obtain, on the one hand, that

$$(5.17) \quad \sum_{k=0}^{\infty} \left\| \Delta_{t_k}^v 2^k f \right\|_{L_{x,v}^1} \leq C \sum_{j=-1}^2 \sum_{k=0}^{\infty} \left\| \Delta_{2^{\lfloor \frac{k}{1-\beta} \rfloor + j}}^v f \right\|_{L_{x,v}^1} \leq C \|f\|_{\tilde{L}^1(dx; B_{1,1}^0(dv))}.$$

And, on the other hand, we have that

$$(5.18) \quad \begin{aligned} \sum_{k=0}^{\infty} \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{s 2^k}^v g \right\|_{L_{x,v}^1} ds &\leq \sum_{j=-\lfloor \frac{1}{1-\beta} \rfloor - 2}^2 \sum_{k=0}^{\infty} \int_{\frac{t_{k-1}}{2}}^{t_k} \left\| \Delta_{2^{\lfloor \frac{k}{1-\beta} \rfloor + j}}^v \Delta_{s 2^k}^v g \right\|_{L_{x,v}^1} ds \\ &\leq C \sum_{j=-\lfloor \frac{1}{1-\beta} \rfloor - 2}^2 \sum_{k=0}^{\infty} \left(t_k - \frac{t_{k-1}}{2} \right) \left\| \Delta_{2^{\lfloor \frac{k}{1-\beta} \rfloor + j}}^v g \right\|_{L_{x,v}^1} \\ &\leq C \sum_{j=-\lfloor \frac{1}{1-\beta} \rfloor - 2}^2 \sum_{k=0}^{\infty} \left(2^{\lfloor \frac{k}{1-\beta} \rfloor + j} \right)^{\beta} \left\| \Delta_{2^{\lfloor \frac{k}{1-\beta} \rfloor + j}}^v g \right\|_{L_{x,v}^1} \\ &\leq C \|g\|_{\tilde{L}^1(dx; B_{1,1}^{\beta}(dv))}. \end{aligned}$$

Finally, combining (5.15) with (5.17) and (5.18) yields

$$(5.19) \quad \left\| \int_{\mathbb{R}^D} f(x, v) dv \right\|_{B_{1,1}^0(dx)} \leq C \left(\|f\|_{\tilde{L}^1(dx; B_{1,1}^0(dv))} + \|g\|_{\tilde{L}^1(dx; B_{1,1}^{\beta}(dv))} \right),$$

where the constant $C > 0$ only depends on fixed parameters, which concludes our proof. \square

Proof of Theorem 5.2. Notice first that there exists $\beta < 1$ so that

$$(5.20) \quad (1 - \Delta_v)^{\frac{\gamma}{2}} g_n \in L^1 \left(\mathbb{R}_x^D; B_{1,1}^{\beta} \left(\mathbb{R}_v^D \right) \right).$$

We may therefore straightforwardly apply Proposition 5.3 to infer that

$$(5.21) \quad \left\{ \int_{\mathbb{R}^D} f_n(x, v) dv \right\}_{n=0}^{\infty} \text{ is bounded in } B_{1,1}^0 \left(\mathbb{R}_x^D \right).$$

Let us recall now a few basic facts from the theory of functions spaces. We refer the reader to [22] or [27] for more details on the subject. The norm that defines the Triebel-Lizorkin spaces $F_{p,q}^s(\mathbb{R}^D)$, for any $1 \leq p, q < \infty$ and $s \in \mathbb{R}$, is given by (in the notation of Section 2.1)

$$(5.22) \quad \|f\|_{F_{p,q}^s} = \left(\int_{\mathbb{R}^D} \left(|\Delta_0 f(x)|^q + \sum_{k=0}^{\infty} 2^{ksq} |\Delta_{2^k} f(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

where $f \in \mathcal{S}'(\mathbb{R}^D)$. Thus, it is clear that $B_{1,1}^0 = F_{1,1}^0$ and therefore $B_{1,1}^0$ is continuously embedded in $F_{1,2}^0$, which is nothing but the local Hardy space h_1 (cf. [27, Section 2.3.5]).

Therefore, it holds that

$$(5.23) \quad \left\{ \int_{\mathbb{R}^D} f_n(x, v) dv \right\}_{n=0}^{\infty} \text{ is bounded in } h_1(\mathbb{R}_x^D) \\ \text{and } \{f_n(x, v)\}_{n=0}^{\infty} \text{ is bounded in } L^1(\mathbb{R}_x^D; h_1(\mathbb{R}_v^D)).$$

We recall now Stein's $L \log L$ result (cf. [24], [25, Chapter 1, §5.2] and [26, Chapter I, §8.14, Chapter III, §5.3]), which states that a sequence of positive functions uniformly bounded in h_1 necessarily satisfies locally a uniform $L \log L$ bound. Here, for a given compact subset $K \subset \mathbb{R}^D$, the Orlicz space $L \log L(K)$ is defined as the Banach space endowed with the norm

$$(5.24) \quad \|f\|_{L \log L(K)} = \inf \left\{ \lambda > 0 : \int_K h\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\},$$

where $h(z) = (1+z) \log(1+z) - z$ is a convex function. Thus, we conclude that, for any compact subsets $K_x, K_v \subset \mathbb{R}^D$,

$$(5.25) \quad \left\{ \int_{\mathbb{R}^D} f_n(x, v) dv \right\}_{n=0}^{\infty} \text{ is bounded in } L \log L(K_x) \\ \text{and } \{f_n(x, v)\}_{n=0}^{\infty} \text{ is bounded in } L^1(\mathbb{R}_x^D; L \log L(K_v)).$$

It then follows that

$$(5.26) \quad \left\{ \int_{\mathbb{R}^D} f_n(x, v) dv \right\}_{n=0}^{\infty} \text{ is equi-integrable} \\ \text{and } \{f_n(x, v)\}_{n=0}^{\infty} \text{ is equi-integrable in } v,$$

and this is, together with the positiveness of the sequences, exactly what is required by Lemma 5.2 of [15] in order to deduce that

$$(5.27) \quad \{f_n(x, v)\}_{n=0}^{\infty} \text{ is equi-integrable (in all variables } x \text{ and } v),$$

which concludes the proof of the theorem. \square

We provide now simple extensions of Theorem 5.2 and Proposition 5.3 to the time dependent transport equation with a vanishing temporal derivative. These variants of the preceding results are precisely the versions needed for their application to the viscous incompressible hydrodynamic limit of the Boltzmann equation without cutoff in [2]. Moreover, the proofs provided below show how to smoothly adapt the general methods developed in this work to the time dependent kinetic transport equation.

Theorem 5.4. *Let $\{f_n(t, x, v)\}_{n=0}^{\infty}$ be a bounded sequence in $L^{\infty}(\mathbb{R}_t; L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D))$ and in $L^1(\mathbb{R}_t \times \mathbb{R}_x^D; B_{1,1}^0(\mathbb{R}_v^D))$, such that $f_n \geq 0$ and that*

$$(5.28) \quad (\delta_n \partial_t + v \cdot \nabla_x) f_n = (1 - \Delta_v)^{\frac{\gamma}{2}} g_n$$

for some sequence $\{g_n(t, x, v)\}_{n=0}^{\infty}$ bounded in $L^1(\mathbb{R}_t \times \mathbb{R}_x^D \times \mathbb{R}_v^D)$, some $\gamma \in \mathbb{R}$ and where the sequence $\delta_n > 0$ vanishes as $n \rightarrow \infty$.

Then,

$$(5.29) \quad \{f_n\}_{n=0}^{\infty} \text{ is equi-integrable (in all variables } t, x \text{ and } v).$$

As before, the above theorem will be obtained as a corollary of the following proposition.

Proposition 5.5. *Let $\{f_n(t, x, v)\}_{n=0}^\infty$ be a bounded sequence in $L^1(\mathbb{R}_t \times \mathbb{R}_x^D; B_{1,1}^0(\mathbb{R}_v^D))$ such that*

$$(5.30) \quad (\delta_n \partial_t + v \cdot \nabla_x) f_n = g_n$$

for some bounded sequence $\{g_n(x, v)\}_{n=0}^\infty$ in $L^1(\mathbb{R}_t \times \mathbb{R}_x^D; B_{1,1}^\beta(\mathbb{R}_v^D))$, where $\beta \in \mathbb{R}$ and the sequence $\delta_n > 0$ vanishes as $n \rightarrow \infty$.

Then,

$$(5.31) \quad \left\{ \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\}_{n=0}^\infty \text{ is bounded in } L^1(\mathbb{R}_t; B_{1,1}^0(\mathbb{R}_x^D)).$$

Proof. We give here a short demonstration of this lemma by following closely the proof of Proposition 5.3 and emphasizing the necessary changes. Thus, we first introduce an interpolation parameter $s \in \mathbb{R}$. Then, it trivially holds that

$$(5.32) \quad \begin{cases} (\partial_s + \delta_n \partial_t + v \cdot \nabla_x) f_n = g_n, \\ f_n(s=0) = f_n. \end{cases}$$

Hence the interpolation formula

$$(5.33) \quad f_n(t, x, v) = f_n(t - \delta_n s, x - sv, v) + \int_0^s g_n(t - \delta_n \sigma, x - \sigma v, v) d\sigma.$$

It follows that, in virtue of (4.7),

$$(5.34) \quad \begin{aligned} \Delta_{2^k}^x \int_{\mathbb{R}^D} f_n(t, x, v) dv &= \int_{\mathbb{R}^D} (\Delta_{s2^k}^v f_n)(t - \delta_n s, x - sv, v) dv \\ &\quad + \int_0^s \int_{\mathbb{R}^D} (\Delta_{\sigma 2^k}^v g_n)(t - \delta_n \sigma, x - \sigma v, v) dv d\sigma, \end{aligned}$$

which substitutes the interpolation formula (4.8).

Then, integrating in time and space yields

$$(5.35) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\|_{L_{t,x}^1} \leq \left\| \Delta_{s2^k}^v f_n \right\|_{L_{t,x,v}^1} + \int_0^s \left\| \Delta_{\sigma 2^k}^v g_n \right\|_{L_{t,x,v}^1} d\sigma.$$

For each value of k , we fix now the interpolation parameter s as $s_k = 2^k \frac{\beta}{1-\beta}$, which yields, summing over k ,

$$(5.36) \quad \sum_{k=0}^\infty \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\|_{L_{t,x}^1} \leq \sum_{k=0}^\infty \left\| \Delta_{s_k 2^k}^v f_n \right\|_{L_{t,x,v}^1} + \sum_{k=0}^\infty \int_0^{s_k} \left\| \Delta_{\sigma 2^k}^v g_n \right\|_{L_{t,x,v}^1} d\sigma.$$

The remainder of the demonstration only consists in an analysis of the velocity frequencies and it is thus strictly identical to the proof of Proposition 5.3. And so, we infer that

$$(5.37) \quad \left\| \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\|_{L^1(dt; B_{1,1}^0(dx))} \leq C \left(\|f_n\|_{L^1(dt dx; B_{1,1}^0(dv))} + \|g_n\|_{L^1(dt dx; B_{1,1}^\beta(dv))} \right),$$

which concludes our proof. \square

Proof of Theorem 5.4. Notice first that there exists $\beta < 1$ so that

$$(5.38) \quad (1 - \Delta_v)^{\frac{\gamma}{2}} g_n \in L^1 \left(\mathbb{R}_t \times \mathbb{R}_x^D; B_{1,1}^\beta \left(\mathbb{R}_v^D \right) \right).$$

We may therefore straightforwardly apply Proposition 5.5 to infer that

$$(5.39) \quad \left\{ \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\}_{n=0}^\infty \text{ is bounded in } L^1 \left(\mathbb{R}_t; B_{1,1}^0 \left(\mathbb{R}_x^D \right) \right).$$

Then, just as in the proof of lemma 5.2, using Stein's $L \log L$ result, we conclude that, for any compact subsets $K_x, K_v \subset \mathbb{R}^D$,

$$(5.40) \quad \left\{ \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\}_{n=0}^\infty \text{ is bounded in } L^1(\mathbb{R}_t; L \log L(K_x))$$

and $\{f_n(t, x, v)\}_{n=0}^\infty$ is bounded in $L^1(\mathbb{R}_t \times \mathbb{R}_x^D; L \log L(K_v))$.

Since $\{f_n(t, x, v)\}_{n=0}^\infty$ is also bounded in $L^\infty(\mathbb{R}_t; L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D))$, it then follows that

$$(5.41) \quad \left\{ \int_{\mathbb{R}^D \times \mathbb{R}^D} f_n(t, x, v) dx dv \right\}_{n=0}^\infty \text{ is equi-integrable in } t,$$

$$\left\{ \int_{\mathbb{R}^D} f_n(t, x, v) dv \right\}_{n=0}^\infty \text{ is equi-integrable in } x$$

and $\{f_n(t, x, v)\}_{n=0}^\infty$ is equi-integrable in v .

By the positiveness of the sequences, it is then possible to apply here Lemma 5.2 of [15] iteratively (one may also consult the proof of Proposition 3.3.5 of [23]) in order to conclude that

$$(5.42) \quad \{f_n(t, x, v)\}_{n=0}^\infty \text{ is equi-integrable (in all variables } t, x \text{ and } v),$$

which concludes the proof of the theorem. \square

6. PROOFS OF THE MAIN RESULTS

We proceed now to the demonstrations of the main results from Section 3. Since all our proofs are based on the crucial decomposition (4.21) given by Proposition 4.1, we will systematically start the proofs by establishing sharp dispersive estimates on the operators A_δ^t and B_δ^t appearing in the right-hand side of (4.21). In accordance with the hypotheses of Proposition 4.1, we will always consider the operators A_δ^t and B_δ^t as defined for some given cutoff $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(s) ds = \int_{\mathbb{R}} \tilde{\rho}(s) ds = 1$ and that $\tilde{\rho}$ is compactly supported inside $[1, 2]$.

6.1. Velocity averaging in $L_x^1 L_v^p$, inhomogeneous case. In this section, we show the proofs of the averaging lemmas in the endpoint case $L_x^1 L_v^p$, i.e. of Theorems 3.1 and 3.2.

For the sake of simplicity, we will only consider here the operators A_δ^t and B_δ^t as defined for the specific cutoff $\tilde{\rho}(s) = \delta_{\{s=1\}}$, so that

$$(6.1) \quad A_\delta^t f(x) = \int_{\mathbb{R}^D} \Delta_\delta^x f(x - tv, v) dv,$$

$$B_\delta^t g(x) = \int_0^1 \int_{\mathbb{R}^D} \Delta_\delta^x g(x - \sigma tv, v) dv d\sigma.$$

This simplification is performed at absolutely no cost of generality, since it is then possible to easily extend the results to the case of a general smooth cutoff $\rho \in \mathcal{S}(\mathbb{R})$ by considering

$$(6.2) \quad \int_{\mathbb{R}} [A_{\delta}^{st} f(x)] \tilde{\rho}(s) ds \quad \text{and} \quad \int_{\mathbb{R}} [B_{\delta}^{st} g(x)] s \tilde{\rho}(s) ds.$$

It is to be emphasized that the above simplification is possible only because the identities (4.23) from Proposition 4.1 will not be employed here.

Lemma 6.1. *For every $1 \leq p, q \leq \infty$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $t \geq 2^{-k}$, it holds that*

$$(6.3) \quad \|A_0^1 f\|_{L_x^p} \leq C \|\Delta_0^v f\|_{L_x^1 L_v^p} \leq C \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^{\alpha}(\mathbb{R}_v^D))}$$

and

$$(6.4) \quad \begin{aligned} \|A_{2^k}^t f\|_{L_x^p} &\leq \frac{C}{t^{D(1-\frac{1}{p})}} \|\Delta_{2^k}^v f\|_{L_x^1 L_v^p} \\ &\leq \frac{C}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^{\alpha}(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of t and 2^k .

Furthermore, if

$$(6.5) \quad D \left(1 - \frac{1}{p}\right) < 1,$$

then, for any $\beta \in \mathbb{R}$, it holds that

$$(6.6) \quad \|B_0^1 g\|_{L_x^p} \leq C \|S_2^v g\|_{L_x^1 L_v^p} \leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))}$$

and, if further $\beta + D \left(1 - \frac{1}{p}\right) \neq 1$,

$$(6.7) \quad \begin{aligned} \|B_{2^k}^t g\|_{L_x^p} &\leq \frac{C}{t^{\beta+D(1-\frac{1}{p})} 2^{k\beta}} \left(2^{-j_k(1-\beta-D(1-\frac{1}{p}))} \|\Delta_0^v g\|_{L_x^1 L_v^p} \right. \\ &\quad \left. + \sum_{j=0}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right) \\ &\leq \frac{C}{t^{\gamma+D(1-\frac{1}{p})} 2^{k\gamma}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min \left\{ \beta, 1 - D \left(1 - \frac{1}{p}\right) \right\}$, $j_k \geq 1$ is the largest integer such that $2^{j_k-1} \leq t 2^k$ and $C > 0$ is independent of t and 2^k .

Finally, if $\beta + D \left(1 - \frac{1}{p}\right) = 1$, then

$$(6.8) \quad \begin{aligned} \|B_{2^k}^t g\|_{L_x^p} &\leq \frac{C}{t 2^{k\beta}} \left(\|\Delta_0^v g\|_{L_x^1 L_v^p} + \sum_{j=0}^{j_k+1} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right) \\ &\leq \frac{C}{t 2^{k\beta}} \log \left(1 + t 2^k \right)^{\frac{1}{q}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of t and 2^k .

Proof. These bounds will follow from the dispersive properties of the transport operator, which were first established by Castella and Perthame in [8]. More precisely, we will utilize the following dispersive estimate, valid for any $1 \leq p \leq \infty$, which is a simple consequence of the change of variables $v \mapsto y = x - tv$,

$$\begin{aligned}
 \|h(x - tv, v)\|_{L_x^p L_v^1} &= t^{-D} \left\| h\left(y, \frac{x-y}{t}\right) \right\|_{L_x^p L_y^1} \\
 (6.9) \quad &\leq t^{-D} \left\| h\left(y, \frac{x-y}{t}\right) \right\|_{L_y^1 L_x^p} \\
 &= t^{-D(1-\frac{1}{p})} \|h(x, v)\|_{L_x^1 L_v^p}.
 \end{aligned}$$

Thus, we easily deduce, utilizing identities (4.7), that

$$(6.10) \quad \|A_0^1 f(x)\|_{L_x^p} \leq C \|(\Delta_0^v f)(x, v)\|_{L_x^1 L_v^p} \leq C \|f(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))},$$

and

$$\begin{aligned}
 (6.11) \quad \|A_{2^k}^t f(x)\|_{L_x^p} &\leq C t^{-D(1-\frac{1}{p})} \|(\Delta_{t2^k}^v f)(x, v)\|_{L_x^1 L_v^p} \\
 &\leq C t^{-\alpha-D(1-\frac{1}{p})} 2^{-k\alpha} \|f(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))},
 \end{aligned}$$

which concludes the estimate for A_0^1 and $A_{2^k}^t$.

As for B_0^1 and $B_{2^k}^t$, we first obtain

$$(6.12) \quad \|B_0^1 g(x)\|_{L_x^p} \leq C \int_0^1 s^{-D(1-\frac{1}{p})} \| (S_s^v g)(x, v) \|_{L_x^1 L_v^p} ds$$

and

$$(6.13) \quad \|B_{2^k}^t g(x)\|_{L_x^p} \leq C \int_0^1 (st)^{-D(1-\frac{1}{p})} \|(\Delta_{st2^k}^v g)(x, v)\|_{L_x^1 L_v^p} ds.$$

Then, since $D(1 - \frac{1}{p}) < 1$ and $S_s^v = S_s^v S_2^v$, we deduce that

$$(6.14) \quad \|B_0^1 g(x)\|_{L_x^p} \leq C \| (S_2^v g)(x, v) \|_{L_x^1 L_v^p} \leq C \|g(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

which concludes the estimate on the low frequency component.

Regarding the high frequencies, we further split the integral in (6.13) into small dyadic intervals

$$\begin{aligned}
 (6.15) \quad I_0 &= \left[0, \frac{1}{t2^k}\right], \quad I_1 = \left[\frac{1}{t2^k}, \frac{2}{t2^k}\right], \quad \dots, \quad I_j = \left[\frac{2^{j-1}}{t2^k}, \frac{2^j}{t2^k}\right] \\
 &\text{and finally } I_{j_k} = \left[\frac{2^{j_k-1}}{t2^k}, 1\right],
 \end{aligned}$$

where $j_k \geq 1$ is the largest integer such that $2^{j_k-1} \leq t2^k$. Thus, on each dyadic interval I_j , where $1 \leq j \leq j_k$, the frequency $st2^k$ is between 2^{j-1} and 2^j . Therefore,

we deduce that, for any $1 \leq j \leq j_k$,

$$\begin{aligned}
 (6.16) \quad \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds &= \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &\leq C |I_j| 2^{-(j-k)D(1-\frac{1}{p})} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \mathcal{G} \right\|_{L_x^1 L_v^p} \\
 &= \frac{C}{t} 2^{(j-k)(1-D(1-\frac{1}{p}))} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \mathcal{G} \right\|_{L_x^1 L_v^p}.
 \end{aligned}$$

Furthermore, when $j = 0$, it is readily seen, since $D(1 - \frac{1}{p}) < 1$, that

$$\begin{aligned}
 (6.17) \quad \int_{I_0} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds &= \int_{I_0} (st)^{-D(1-\frac{1}{p})} \left\| S_4^v \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &\leq C \int_{I_0} (st)^{-D(1-\frac{1}{p})} ds \|S_4^v \mathcal{G}\|_{L_x^1 L_v^p} \\
 &= C \frac{1}{t(1-D(1-\frac{1}{p}))} 2^{-k(1-D(1-\frac{1}{p}))} \|S_4^v \mathcal{G}\|_{L_x^1 L_v^p}.
 \end{aligned}$$

Thus, on the whole, we have obtained that

$$\begin{aligned}
 (6.18) \quad &\|B_{2^k}^t \mathcal{G}\|_{L_x^p} \\
 &\leq C \int_0^1 (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &= C \sum_{j=0}^{j_k} \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &\leq \frac{C}{t} \left(2^{-k(1-D(1-\frac{1}{p}))} \|\Delta_0^v \mathcal{G}\|_{L_x^1 L_v^p} + \sum_{j=0}^{j_k+1} 2^{(j-k)(1-D(1-\frac{1}{p}))} \|\Delta_{2^j}^v \mathcal{G}\|_{L_x^1 L_v^p} \right) \\
 &\leq Ct^{-\beta-D(1-\frac{1}{p})} 2^{-k\beta} \left(2^{-j_k(1-\beta-D(1-\frac{1}{p}))} \|\Delta_0^v \mathcal{G}\|_{L_x^1 L_v^p} \right. \\
 &\quad \left. + \sum_{j=0}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v \mathcal{G}\|_{L_x^1 L_v^p} \right).
 \end{aligned}$$

It follows that, if $1 > \beta + D(1 - \frac{1}{p})$, a further application of Hölder's inequality to the preceding estimate yields

$$\begin{aligned}
 (6.19) \quad &\|B_{2^k}^t \mathcal{G}(x)\|_{L_x^p} \\
 &\leq Ct^{-\beta-D(1-\frac{1}{p})} 2^{-k\beta} \left\| \left\{ 2^{-j(1-\beta-D(1-\frac{1}{p}))} \right\}_{j=0}^{\infty} \right\|_{\ell^{q'}} \|g(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}.
 \end{aligned}$$

And in case $1 < \beta + D \left(1 - \frac{1}{p}\right)$, we obtain

$$(6.20) \quad \begin{aligned} & \|B_{2^k}^t g(x)\|_{L_x^p} \\ & \leq \frac{C}{t} 2^{k(D(1-\frac{1}{p})-1)} \left\| \left\{ 2^{j(1-\beta-D(1-\frac{1}{p}))} \right\}_{j=0}^{\infty} \right\|_{\ell^{q'}} \|g(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Finally, if $1 = \beta + D \left(1 - \frac{1}{p}\right)$, since $j_k + 3 \leq \frac{\log(t2^{k+4})}{\log 2}$, then

$$(6.21) \quad \begin{aligned} \|B_{2^k}^t g\|_{L_x^p} & \leq \frac{C}{t2^{k\beta}} \left(\|\Delta_0^v g\|_{L_x^1 L_v^p} + \sum_{j=0}^{j_k+1} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right) \\ & \leq \frac{C}{t2^{k\beta}} (j_k + 3)^{\frac{1}{q'}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \\ & \leq \frac{C}{t2^{k\beta}} \log(1 + t2^k)^{\frac{1}{q'}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

which concludes the proof of the lemma. \square

We proceed now to the proof of Theorem 3.1, which is the simplest of our main results.

Proof of Theorem 3.1. According to Proposition 4.1, we begin with the following dyadic interpolation formulas, for each $k \in \mathbb{N}$,

$$(6.22) \quad \begin{aligned} \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv &= A_0^1 f(x) + B_0^1 g(x), \\ \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv &= A_{2^k}^t f(x) + t B_{2^k}^t g(x). \end{aligned}$$

Then, a direct application of Lemma 6.1 yields the estimates, for every $t \geq 2^{-k}$,

$$(6.23) \quad \begin{aligned} \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} & \leq C \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} + C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \\ \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} & \leq \frac{C}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} (t2^k)^\alpha \|\Delta_{t2^k}^v f\|_{L_x^1 L_v^p} \\ & \quad + \frac{C}{t^{\beta-1+D(1-\frac{1}{p})} 2^{k\beta}} \left(2^{-j_k(1-\beta-D(1-\frac{1}{p}))} \|\Delta_0^v g\|_{L_x^1 L_v^p} \right. \\ & \quad \left. + \sum_{j=0}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right), \end{aligned}$$

where $j_k \geq 1$ is the largest integer such that $2^{j_k-1} \leq t2^k$ and $C > 0$ is independent of t and 2^k , which concludes the estimate on the low frequencies of the velocity averages $\Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv$. Note that the above estimate remains valid for the value $\beta + D \left(1 - \frac{1}{p}\right) = 1$.

Next, in order to treat the high frequencies, for each value of k , we have to select the value of the interpolation parameter t which will optimize (6.23), i.e. minimize its right-hand side. The heuristic argument yielding the optimal value for t goes

as follows. In virtue of Lemma 6.1, the estimate on the high frequencies in (6.23) essentially amounts to

$$(6.24) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \leq C \left(\frac{1}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} + \frac{1}{t^{\gamma-1+D(1-\frac{1}{p})} 2^{k\gamma}} \right),$$

where $\gamma = \min \left\{ \beta, 1 - D \left(1 - \frac{1}{p} \right) \right\}$. Therefore, up to multiplicative constants, the right-hand side above will be minimized when both terms are equal, which leads to an optimal value of the interpolation parameter t of

$$(6.25) \quad t_k = 2^{-k \frac{\alpha-\gamma}{1+\alpha-\gamma}} \geq 2^{-k},$$

where we have used the hypothesis $\alpha > -D \left(1 - \frac{1}{p} \right)$ to deduce that $1 + \alpha - \gamma > 0$.

Note that, in the case $\beta + D \left(1 - \frac{1}{p} \right) \geq 1$, we can choose $t = \infty$, which is, in fact, more optimal than $t 2^k = 2^{k \frac{1}{1+\alpha-\gamma}}$, since it eliminates the first term in the right-hand side of the above estimates. This case is discussed later on.

Thus, in the case $\beta + D \left(1 - \frac{1}{p} \right) < 1$ so that $\gamma = \beta$, setting $t = t_k$ in (6.23), denoting $s = \frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\gamma} - D \left(1 - \frac{1}{p} \right)$, noticing that $j_k = \left\lceil 1 + \frac{k}{1+\alpha-\gamma} \right\rceil$ and summing over k , yields that

$$(6.26) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ \left(t_k 2^k \right)^\alpha \left\| \Delta_{t_k 2^k}^v f \right\|_{L_x^1 L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & + C \left\| \left\{ 2^{-k \frac{1}{1+\alpha-\beta} (1-\beta-D(1-\frac{1}{p}))} \right\}_{k=0}^\infty \right\|_{\ell^q} \left\| \Delta_0^v g \right\|_{L_x^1 L_v^p} \\ & + C \left\| \left\{ \sum_{j=0}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q}. \end{aligned}$$

Then, it is readily seen that the first term in the right-hand side above is controlled by $\|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))}$, for $t_k 2^k = 2^{k \frac{1}{1+\alpha-\beta}} \rightarrow \infty$ as $k \rightarrow \infty$. Furthermore, writing $a_j = 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p} \mathbb{1}_{\{j \geq 0\}}$ and $b_j = 2^{-j(1-\beta-D(1-\frac{1}{p}))} \mathbb{1}_{\{j \geq 0\}}$, for all $j \in \mathbb{Z}$, we see that $a = \{a_j\}_{j \in \mathbb{Z}} \in \ell^q$, $b = \{b_j\}_{j \in \mathbb{Z}} \in \ell^1$ and that

$$(6.27) \quad \begin{aligned} & \left\| \left\{ \sum_{j=0}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & = \left\| \left\{ (a * b)_{j_k+1} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq \|a\|_{\ell^q} \|b\|_{\ell^1} \\ & \leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Thus, on the whole, combining the preceding estimates with (6.26), we deduce that

$$(6.28) \quad \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \leq C \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} + C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

which concludes the proof in the case $\beta + D \left(1 - \frac{1}{p}\right) < 1$.

Regarding the case $\beta + D \left(1 - \frac{1}{p}\right) \geq 1$ so that $\gamma = 1 - D \left(1 - \frac{1}{p}\right)$, we note that the estimate (6.23) on the high frequencies implies that, recalling j_k satisfies $t2^k \leq 2^{j_k} \leq t2^{k+1}$,

$$(6.29) \quad \begin{aligned} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} &\leq \frac{C}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \\ &+ \frac{C}{2^{k(1-D(1-\frac{1}{p}))}} \left(\|\Delta_0^v g\|_{L_x^1 L_v^p} \right. \\ &\left. + \sum_{j=0}^{j_k+1} 2^{j(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right). \end{aligned}$$

Therefore, letting t tend to infinity, denoting $s = \frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\gamma} - D \left(1 - \frac{1}{p}\right) = 1 - D \left(1 - \frac{1}{p}\right)$ and noticing that $\lim_{t \rightarrow \infty} j_k = \infty$, we find

$$(6.30) \quad \begin{aligned} &2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \\ &\leq C \left(\|\Delta_0^v g\|_{L_x^1 L_v^p} + \sum_{j=0}^\infty 2^{j(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \right). \end{aligned}$$

It then follows that

$$(6.31) \quad \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^\infty} \leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,1}^\beta(\mathbb{R}_v^D))},$$

and, in case $\beta + D \left(1 - \frac{1}{p}\right) > 1$,

$$(6.32) \quad \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^\infty} \leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

which concludes the proof of the theorem. \square

Next is the demonstration of Theorem 3.2, which builds upon the previous proof of Theorem 3.1.

Proof of Theorem 3.2. As in the proof of Theorem 3.1, we begin with the dyadic decomposition provided by Proposition 4.1, for each $k \in \mathbb{N}$,

$$(6.33) \quad \begin{aligned} \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv &= A_0^1 f(x) + B_0^1 g(x), \\ \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv &= A_{2^k}^t f(x) + t B_{2^k}^t g(x), \end{aligned}$$

and we utilize Lemma 6.1 to obtain, in virtue of property (4.25) from Proposition 4.1, that

$$(6.34) \quad \begin{aligned} \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} &\leq C \|S_2^x f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} + C \|S_2^x g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \\ \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} &\leq C \frac{1}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad + C \left[\begin{array}{ll} \frac{t^{1-\gamma-D(1-\frac{1}{p})}}{2^{k\gamma}} & \text{if } \beta + D(1 - \frac{1}{p}) < 1 \\ \frac{\log(1+t2^k)}{2^{k\gamma}} \frac{1}{q'} & \text{if } \beta + D(1 - \frac{1}{p}) = 1 \\ \frac{1}{2^{k\gamma}} & \text{if } \beta + D(1 - \frac{1}{p}) > 1 \end{array} \right] \\ &\quad \times \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min \left\{ \beta, 1 - D \left(1 - \frac{1}{p} \right) \right\}$. This concludes the estimate on the low frequencies of the velocity averages $\Delta_0^x \int_{\mathbb{R}^D} f(x, v) dv$.

Next, in order to control the high frequencies, optimizing in t for each value of k , we fix the interpolation parameter t as $t_k = 2^{-k \frac{(a-\gamma)+(a-b)}{1+\alpha-\gamma}}$ (cf. proof of Theorem 3.1 for a heuristic explanation on how to choose this optimal parameter). Note that $t_k \geq 2^{-k}$, for $b \geq a - 1$ and $1 + \alpha - \gamma > 0$, and that this choice is independent of $1 \leq p, q \leq \infty$.

Furthermore, in the cases $\beta + D \left(1 - \frac{1}{p} \right) > 1$ or $\beta + D \left(1 - \frac{1}{p} \right) = 1$ and $q = 1$, we can choose $t = \infty$, which is, in fact, more optimal than $t2^k = 2^{k \frac{1+b-a}{1+\alpha-\gamma}}$, since it eliminates the first term in the right-hand side of the above estimate. This case is discussed later on.

Therefore, denoting $s = (1 + b - a) \frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\gamma} + a - D \left(1 - \frac{1}{p} \right)$, we find that

$$(6.35) \quad \begin{aligned} &2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \\ &\leq C 2^{ka} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad + C \left[\begin{array}{ll} 2^{kb} & \text{if } \beta + D \left(1 - \frac{1}{p} \right) \neq 1 \\ 2^{kb} \left(1 + \frac{1+b-a}{1+\alpha-\gamma} k \right) \frac{1}{q'} & \text{if } \beta + D \left(1 - \frac{1}{p} \right) = 1 \end{array} \right] \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Consequently, summing over k , we obtain, in case $\beta + D \left(1 - \frac{1}{p}\right) \neq 1$ or $q = 1$,

$$\begin{aligned}
 (6.36) \quad & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
 & \leq C \left\| \left\{ 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\
 & + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\
 & = C \|f\|_{B_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + C \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)},
 \end{aligned}$$

while, if $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q \neq 1$, we find, for any $\epsilon > 0$,

$$\begin{aligned}
 (6.37) \quad & \left\| \left\{ 2^{k(s-\epsilon)} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
 & \leq C \left\| \left\{ 2^{k(a-\epsilon)} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\
 & + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\
 & = C \|f\|_{B_{1,p,q}^{a-\epsilon,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + C \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}.
 \end{aligned}$$

We handle now the cases $\beta + D \left(1 - \frac{1}{p}\right) > 1$ or $\beta + D \left(1 - \frac{1}{p}\right) = 1$ and $q = 1$, by letting t tend to infinity in (6.34), as mentioned previously. This leads to

$$(6.38) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \leq \frac{C}{2^{k\gamma}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}.$$

Hence, recalling $s = 1 + b - D \left(1 - \frac{1}{p}\right)$ and summing over k yields

$$\begin{aligned}
 (6.39) \quad & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
 & \leq C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\
 & = C \|g\|_{B_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)},
 \end{aligned}$$

which concludes the proof of the theorem. \square

6.2. Velocity averaging in $L_x^1 L_v^p$, homogeneous case. Here, we explain how the proofs of the previous section in the inhomogeneous case are adapted to the homogeneous case and we give justifications of Theorems 3.3 and 3.4.

Again, for the sake of simplicity as in the previous section and without any loss of generality, we will only consider here the operators A_δ^t and B_δ^t as defined for the specific cutoff $\tilde{\rho}(s) = \delta_{\{s=1\}}$

Lemma 6.2. *For every $1 \leq p, q \leq \infty$, $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}$ and $t > 0$ such that*

$$(6.40) \quad \beta + D \left(1 - \frac{1}{p} \right) < 1,$$

it holds that

$$(6.41) \quad \begin{aligned} \|A_{2^k}^t f\|_{L_x^p} &\leq \frac{C}{t^{D(1-\frac{1}{p})}} \|\Delta_{t2^k}^v f\|_{L_x^1 L_v^p} \\ &\leq \frac{C}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))}, \end{aligned}$$

and

$$(6.42) \quad \begin{aligned} \|B_{2^k}^t g\|_{L_x^p} &\leq \frac{C}{t^{\beta+D(1-\frac{1}{p})} 2^{k\beta}} \sum_{j=-\infty}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \\ &\leq \frac{C}{t^{\beta+D(1-\frac{1}{p})} 2^{k\beta}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $j_k \geq 1$ is the largest integer such that $2^{j_k-1} \leq t2^k$ and $C > 0$ is independent of t and 2^k .

Furthermore, if $\beta + D \left(1 - \frac{1}{p} \right) = 1$ and $q = 1$, then it still holds that

$$(6.43) \quad \begin{aligned} \|B_{2^k}^t g\|_{L_x^p} &\leq \frac{C}{t2^{k\beta}} \sum_{j=-\infty}^{j_k+1} 2^{j\beta} \|\Delta_{2^j}^v g\|_{L_x^1 L_v^p} \\ &\leq \frac{C}{t2^{k\beta}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,1}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Proof. We follow precisely the steps of the proof of Lemma 6.1. Thus, we first have, as a consequence of the dispersive properties of the transport operator, that

$$(6.44) \quad \begin{aligned} \|A_{2^k}^t f(x)\|_{L_x^p} &\leq Ct^{-D(1-\frac{1}{p})} \left\| \left(\Delta_{t2^k}^v f \right) (x, v) \right\|_{L_x^1 L_v^p} \\ &\leq Ct^{-\alpha-D(1-\frac{1}{p})} 2^{-k\alpha} \|f(x, v)\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))}, \end{aligned}$$

which concludes the estimate for $A_{2^k}^t$.

As for $B_{2^k}^t$, we first obtain

$$(6.45) \quad \|B_{2^k}^t g(x)\|_{L_x^p} \leq C \int_0^1 (st)^{-D(1-\frac{1}{p})} \left\| \left(\Delta_{st2^k}^v g \right) (x, v) \right\|_{L_x^1 L_v^p} ds.$$

Then, we further split the above integral into small dyadic intervals

$$(6.46) \quad I_j = \left[\frac{2^{j-1}}{t2^k}, \frac{2^j}{t2^k} \right] \quad \text{and} \quad I_{j_k} = \left[\frac{2^{j_k-1}}{t2^k}, 1 \right],$$

where $j, j_k \in \mathbb{Z}$, $j \leq j_k$ and j_k is the largest integer such that $2^{j_k-1} \leq t2^k$. Thus, on each dyadic interval I_j , the frequency $st2^k$ is between 2^{j-1} and 2^j . Therefore, we

deduce that

$$\begin{aligned}
 (6.47) \quad \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds &= \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &\leq C |I_j| 2^{-(j-k)D(1-\frac{1}{p})} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \mathcal{G} \right\|_{L_x^1 L_v^p} \\
 &= \frac{C}{t} 2^{(j-k)(1-D(1-\frac{1}{p}))} \left\| \Delta_{[2^{j-2}, 2^{j+1}]}^v \mathcal{G} \right\|_{L_x^1 L_v^p}.
 \end{aligned}$$

Thus, on the whole, we have obtained that

$$\begin{aligned}
 (6.48) \quad \|B_{2^k}^t \mathcal{G}\|_{L_x^p} &\leq C \int_0^1 (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &= C \sum_{j=-\infty}^{j_k} \int_{I_j} (st)^{-D(1-\frac{1}{p})} \left\| \Delta_{st2^k}^v \mathcal{G} \right\|_{L_x^1 L_v^p} ds \\
 &\leq \frac{C}{t} \sum_{j=-\infty}^{j_k+1} 2^{(j-k)(1-D(1-\frac{1}{p}))} \left\| \Delta_{2^j}^v \mathcal{G} \right\|_{L_x^1 L_v^p} \\
 &\leq C t^{-\beta-D(1-\frac{1}{p})} 2^{-k\beta} \sum_{j=-\infty}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v \mathcal{G} \right\|_{L_x^1 L_v^p}.
 \end{aligned}$$

It follows that, if $1 - \beta - D(1 - \frac{1}{p}) > 0$, a further application of Hölder's inequality to the preceding estimate yields

$$\begin{aligned}
 (6.49) \quad \|B_{2^k}^t \mathcal{G}(x)\|_{L_x^p} &\leq C t^{-\beta-D(1-\frac{1}{p})} 2^{-k\beta} \left\| \left\{ 2^{-j(1-\beta-D(1-\frac{1}{p}))} \right\}_{j=0}^{\infty} \right\|_{\ell^{q'}} \left\| \mathcal{G}(x, v) \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))},
 \end{aligned}$$

which concludes the proof of the lemma. \square

Proof of Theorem 3.3. According to Proposition 4.1, we begin with the following dyadic interpolation formula, for each $k \in \mathbb{N}$,

$$(6.50) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv = A_{2^k}^t f(x) + t B_{2^k}^t \mathcal{G}(x).$$

Then, a direct application of Lemma 6.2 yields the estimate, for every $t > 0$,

$$\begin{aligned}
 (6.51) \quad &\left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \\
 &\leq \frac{C}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} (t^{2^k})^\alpha \left\| \Delta_{t2^k}^v f \right\|_{L_x^1 L_v^p} \\
 &\quad + \frac{C}{t^{\beta-1+D(1-\frac{1}{p})} 2^{k\beta}} \sum_{j=-\infty}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v \mathcal{G} \right\|_{L_x^1 L_v^p},
 \end{aligned}$$

where $j_k \in \mathbb{Z}$ is the largest integer such that $2^{j_k-1} \leq t^{2^k}$ and $C > 0$ is independent of t and 2^k .

Next, optimizing in t for each value of k , we fix the interpolation parameter t as

$$(6.52) \quad t_k = \lambda^{\frac{1}{1+\alpha-\beta}} 2^{-k \frac{\alpha-\beta}{1+\alpha-\beta}},$$

where $\lambda = \frac{\|f\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))}}{\|g\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}}$ (cf. proof of Theorem 3.1 for a heuristic explanation on how to choose this optimal parameter). Note that $1 + \alpha - \beta > 0$ since $\alpha > -D \left(1 - \frac{1}{p}\right)$. Thus, setting $t = t_k$ in (6.51), denoting $s = \frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta} - D \left(1 - \frac{1}{p}\right)$, noticing that $j_k = \left\lceil 1 + \frac{\log \lambda}{(1+\alpha-\beta) \log 2} + \frac{k}{1+\alpha-\beta} \right\rceil$ and summing over k , yields that

$$(6.53) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ & \leq \frac{C}{\lambda^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}}} \left\| \left\{ (t_k 2^k)^\alpha \left\| \Delta_{t_k 2^k}^v f \right\|_{L_x^1 L_v^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ & + \frac{C}{\lambda^{\frac{\beta-1+D(1-\frac{1}{p})}{1+\alpha-\beta}}} \left\| \left\{ \sum_{j=-\infty}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q}. \end{aligned}$$

Then, it is readily seen that the first term in the right-hand side above is controlled by $\|f\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))}$, for $t_k 2^k = \left(\lambda 2^k\right)^{\frac{1}{1+\alpha-\beta}} \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$. Furthermore, writing $a_j = 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p}$ and $b_j = 2^{-j(1-\beta-D(1-\frac{1}{p}))} \mathbf{1}_{\{j \geq 0\}}$, for all $j \in \mathbb{Z}$, we see that $a = \{a_j\}_{j \in \mathbb{Z}} \in \ell^q$, $b = \{b_j\}_{j \in \mathbb{Z}} \in \ell^1$ and that

$$(6.54) \quad \begin{aligned} & \left\| \left\{ \sum_{j=-\infty}^{j_k+1} 2^{-(j_k+1-j)(1-\beta-D(1-\frac{1}{p}))} 2^{j\beta} \left\| \Delta_{2^j}^v g \right\|_{L_x^1 L_v^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ & = \left\| \{(a * b)_{j_k+1}\}_{k=-\infty}^\infty \right\|_{\ell^q} \leq \|a\|_{\ell^q} \|b\|_{\ell^1} \leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Thus, on the whole, combining the preceding estimates with (6.53), we deduce that

$$(6.55) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ & \leq C \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))}^{\frac{1-\beta-D(1-\frac{1}{p})}{1+\alpha-\beta}} \times \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}}, \end{aligned}$$

which concludes the proof of the theorem. \square

Proof of Theorem 3.4. As in the proof of Theorem 3.3, we begin with the dyadic decomposition provided by Proposition 4.1, for each $k \in \mathbb{N}$,

$$(6.56) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv = A_{2^k}^t f(x) + t B_{2^k}^t g(x),$$

and we utilize Lemma 6.2 to obtain, in virtue of property (4.25) from Proposition 4.1, that

$$(6.57) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \leq C \frac{1}{t^{\alpha+D(1-\frac{1}{p})} 2^{k\alpha}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))} \\ + C \frac{t^{1-\beta-D(1-\frac{1}{p})}}{2^{k\beta}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}.$$

Next, optimizing in t for each value of k , we fix the interpolation parameter t as

$$(6.58) \quad t_k = \lambda^{\frac{1}{1+\alpha-\beta}} 2^{-k \frac{(\alpha-\beta)+(a-b)}{1+\alpha-\beta}},$$

where $\lambda = \frac{\|f\|_{\dot{B}_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}}{\|g\|_{\dot{B}_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}}$ (cf. proof of Theorem 3.1 for a heuristic explanation on how to choose this optimal parameter). Note that $1 + \alpha - \beta > 0$ since $\alpha > -D(1 - \frac{1}{p})$ and that this choice is independent of $1 \leq p, q \leq \infty$. Therefore, denoting $s = (1 + b - a) \frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta} + a - D(1 - \frac{1}{p})$, we find that

$$(6.59) \quad 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \leq \frac{C}{\lambda^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}}} 2^{ka} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))} \\ + \frac{C}{\lambda^{\frac{\beta-1+D(1-\frac{1}{p})}{1+\alpha-\beta}}} 2^{kb} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))}.$$

Finally, summing over k , we obtain

$$(6.60) \quad \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) dv \right\|_{L_x^p} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ \leq \frac{C}{\lambda^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}}} \left\| \left\{ 2^{ka} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ + \frac{C}{\lambda^{\frac{\beta-1+D(1-\frac{1}{p})}{1+\alpha-\beta}}} \left\| \left\{ 2^{kb} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^1(\mathbb{R}_x^D; \dot{B}_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-\infty}^\infty \right\|_{\ell^q} \\ = C \|f\|_{\dot{B}_{1,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{1-\beta-D(1-\frac{1}{p})}{1+\alpha-\beta}} \times \|g\|_{\dot{B}_{1,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{\alpha+D(1-\frac{1}{p})}{1+\alpha-\beta}},$$

which concludes the proof of the theorem. \square

6.3. The classical $L_x^2 L_v^2$ case revisited. We provide now the proofs for the classical Hilbertian case of velocity averaging.

Lemma 6.3. *Let $\phi(v) \in C_0^\infty(\mathbb{R}^D)$. For every $1 \leq q \leq \infty$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq \frac{1}{2}$, $k \in \mathbb{N}$ and $t \geq 2^{-k}$, it holds that*

(6.61)

$$\begin{aligned} \|A_0^1(f\phi)\|_{L_x^2} &\leq C \|\Delta_0^x f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \leq C \|f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))}, \\ \|A_{2^k}^t(f\phi)\|_{L_x^2} &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|\Delta_{2^k}^x f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))}, \end{aligned}$$

and

(6.62)

$$\begin{aligned} \|B_0^1(g\phi)\|_{L_x^2} &\leq C \|\Delta_0^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \leq C \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{(t2^k)^{\gamma+\frac{1}{2}}} \|\Delta_{2^k}^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \leq \frac{C}{(t2^k)^{\gamma+\frac{1}{2}}} \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min\left\{\beta, \frac{1}{2}\right\}$ and $C > 0$ is independent of t and 2^k .

Furthermore, if $\beta = \frac{1}{2}$, then, it holds that

$$\begin{aligned} \|B_0^1(g\phi)\|_{L_x^2} &\leq C \|\Delta_0^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \leq C \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \\ (6.63) \quad \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{t^{2k}} \log\left(1 + t^{2k}\right)^{\frac{1}{q'}} \|\Delta_{2^k}^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \\ &\leq \frac{C}{t^{2k}} \log\left(1 + t^{2k}\right)^{\frac{1}{q'}} \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

The above lemma will be obtained as a consequence of the following Lemmas 6.4 and 6.5.

Lemma 6.4. *For any $\chi \in \mathcal{S}(\mathbb{R})$ and $\rho \in C^\infty(\mathbb{R})$ such that $\chi \equiv 1$ in a neighborhood of the origin and $\hat{\rho}$ is compactly supported and bounded pointwise, and for each $\lambda \geq 1$, $R > 0$ and $t \geq \frac{1}{R}$, let $h_1(\eta, v)$ and $h_2(\eta, v)$ be defined by*

$$\begin{aligned} (6.64) \quad h_1(\eta, v) &= \chi\left(\left|v - \frac{v \cdot \eta}{|\eta|} \frac{\eta}{|\eta|}\right|\right) \rho\left(\lambda \frac{v \cdot \eta}{|\eta|}\right), \\ h_2(\eta, v) &= \mathbb{1}_{\{\frac{R}{2} \leq |\eta| \leq 2R\}} \chi\left(\left|v - \frac{v \cdot \eta}{|\eta|} \frac{\eta}{|\eta|}\right|\right) \rho(tv \cdot \eta), \end{aligned}$$

for all $\eta, v \in \mathbb{R}^D$.

Then, for each $k \in \mathbb{N}$, $\|\Delta_0^v h_1\|_{L_v^2}$ and $\|\Delta_{2^k}^v h_1\|_{L_v^2}$ are independent of η and, for every $1 \leq q \leq \infty$ and $\alpha \neq \frac{1}{2}$, it holds that

$$\begin{aligned} (6.65) \quad \|h_1\|_{B_{2,q}^{-\alpha}(\mathbb{R}_v^D)} &\leq \frac{C}{\lambda^{\gamma+\frac{1}{2}}}, \\ \|h_2\|_{\tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q}^{-\alpha}(\mathbb{R}_v^D))} &\leq \frac{C}{(tR)^{\gamma+\frac{1}{2}}}, \end{aligned}$$

where $\gamma = \min\left\{\alpha, \frac{1}{2}\right\}$ and the constant $C > 0$ is independent of λ , t and R .

Furthermore, if $\alpha = \frac{1}{2}$, then it holds that

$$(6.66) \quad \begin{aligned} \|h_1\|_{B_{2,q}^{-\alpha}(\mathbb{R}_v^D)} &\leq \frac{C}{\lambda} \log(1+\lambda)^{\frac{1}{q}}, \\ \|h_2\|_{\tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q}^{-\alpha}(\mathbb{R}_v^D))} &\leq \frac{C}{tR} \log(1+tR)^{\frac{1}{q}}. \end{aligned}$$

Finally, if the support of $\hat{\rho}$ doesn't contain the origin, then, for every $\alpha \in \mathbb{R}$, it holds that

$$(6.67) \quad \begin{aligned} \|h_1\|_{B_{2,q}^{-\alpha}(\mathbb{R}_v^D)} &\leq \frac{C}{\lambda^{\alpha+\frac{1}{2}}}, \\ \|h_2\|_{\tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q}^{-\alpha}(\mathbb{R}_v^D))} &\leq \frac{C}{(tR)^{\alpha+\frac{1}{2}}}. \end{aligned}$$

Lemma 6.5. For any $\chi \in \mathcal{S}(\mathbb{R})$ and $\rho \in C^\infty(\mathbb{R})$ such that $\chi \equiv 1$ in a neighborhood of the origin and $\hat{\rho}$ is compactly supported and bounded pointwise, let $h(\eta, v)$ be defined by

$$(6.68) \quad h(\eta, v) = \mathbb{1}_{\{|\eta| \leq 1\}} \chi(|v|) \rho(v \cdot \eta),$$

for all $\eta, v \in \mathbb{R}^D$.

Then, for every $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$, it holds that

$$(6.69) \quad h(\eta, v) \in \tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q}^{-\alpha}(\mathbb{R}_v^D)).$$

We defer the proofs of Lemmas 6.4 and 6.5 and proceed now to the proof of Lemma 6.3.

Proof of Lemma 6.3. First, notice that, for any $\rho(s) \in \mathcal{S}(\mathbb{R})$, it holds

$$(6.70) \quad \mathcal{F}\left(\frac{\rho(s) - \rho(0)}{is}\right)(r) = 2\pi\rho(0)\mathbb{1}_{\{r \geq 0\}} - \int_{-\infty}^r \hat{\rho}(\sigma) d\sigma,$$

in the sense of tempered distributions. Indeed, one easily obtains by duality, for any test function $\varphi(r) \in \mathcal{S}(\mathbb{R})$, that

$$(6.71) \quad \begin{aligned} \int_{\mathbb{R}} \mathcal{F}\left(\frac{\rho(s) - \rho(0)}{is}\right)(r) \varphi(r) dr &= \int_{\mathbb{R}} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{is\sigma} - 1}{is} \hat{\rho}(\sigma) d\sigma \hat{\varphi}(s) ds \\ &= \int_{\mathbb{R}} \frac{1}{2\pi} \int_{\mathbb{R}} \left[\int_0^\sigma e^{ist} dt \right] \hat{\rho}(\sigma) d\sigma \hat{\varphi}(s) ds \\ &= \int_{\mathbb{R}} \left[\int_0^\sigma \varphi(t) dt \right] \hat{\rho}(\sigma) d\sigma \\ &= \int_0^\infty \int_t^\infty \varphi(t) \hat{\rho}(\sigma) d\sigma dt - \int_{-\infty}^0 \int_{-\infty}^t \varphi(t) \hat{\rho}(\sigma) d\sigma dt \\ &= \int_0^\infty \int_{\mathbb{R}} \hat{\rho}(\sigma) d\sigma \varphi(t) dt - \int_{\mathbb{R}} \int_{-\infty}^t \hat{\rho}(\sigma) d\sigma \varphi(t) dt \\ &= \int_{\mathbb{R}} \left[2\pi\rho(0)\mathbb{1}_{\{t \geq 0\}} - \int_{-\infty}^t \hat{\rho}(\sigma) d\sigma \right] \varphi(t) dt. \end{aligned}$$

In the particular setting of the present lemma, we impose that $\hat{\rho}$ be compactly supported, supported away from the origin and that $\rho(0) = 1$. It then follows that

$\tau(s) = \frac{1-\rho(s)}{is}$ is smooth near the origin, that

$$(6.72) \quad \hat{\tau}(r) = \int_{\mathbb{R}} e^{-isr} \frac{1-\rho(s)}{is} ds = \int_{-\infty}^r \hat{\rho}(\sigma) d\sigma - 2\pi \mathbb{1}_{\{r \geq 0\}}$$

is compactly supported as well and that $|\hat{\tau}(r)|$ is bounded pointwise. Notice, however, that the origin is always contained in the support of $\hat{\tau}$.

Consider now $R > 0$ so that $\text{supp}\phi(v) \subset \{|v| \leq R\}$ and let $\chi(r) \in C_0^\infty(\mathbb{R})$ be a cutoff function satisfying $\mathbb{1}_{\{|r| \leq R\}} \leq \chi(r) \leq \mathbb{1}_{\{|r| \leq 2R\}}$. Then, according to the identities (4.23) from Proposition 4.1 and Lemmas 6.4 and 6.5, for any $t \geq 2^{-k}$, we obtain

$$(6.73) \quad \begin{aligned} \|A_0^1(f\phi)\|_{L_x^2} &= \frac{1}{(2\pi)^{\frac{D}{2}}} \left\| \int_{\mathbb{R}^D} \psi(\eta) \hat{f}(\eta, v) \phi(v) \chi(|v|) \rho(v \cdot \eta) dv \right\|_{L_\eta^2} \\ &\leq \|\Delta_0^x f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \left\| \mathbb{1}_{\{|\eta| \leq 1\}} \chi(|v|) \rho(v \cdot \eta) \right\|_{\tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q'}^{-\alpha}(\mathbb{R}_v^D))} \\ &\leq C \|\Delta_0^x f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \leq C \|f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))}, \\ \|A_{2^k}^t(f\phi)\|_{L_x^2} &= \frac{1}{(2\pi)^{\frac{D}{2}}} \left\| \int_{\mathbb{R}^D} \varphi\left(\frac{\eta}{2^k}\right) \hat{f}(\eta, v) \phi(v) \chi\left(v - \frac{v \cdot \eta}{|\eta|} \frac{\eta}{|\eta|}\right) \rho(tv \cdot \eta) dv \right\|_{L_\eta^2} \\ &\leq \|\Delta_{2^k}^x f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad \times \left\| \mathbb{1}_{\{2^{k-1} \leq |\eta| \leq 2^{k+1}\}} \chi\left(v - \frac{v \cdot \eta}{|\eta|} \frac{\eta}{|\eta|}\right) \rho(tv \cdot \eta) \right\|_{\tilde{L}^\infty(\mathbb{R}_\eta^D; B_{2,q'}^{-\alpha}(\mathbb{R}_v^D))} \\ &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|\Delta_{2^k}^x f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \\ &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|f\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \end{aligned}$$

and, similarly,

$$(6.74) \quad \begin{aligned} \|B_0^1(g\phi)\|_{L_x^2} &\leq C \|\Delta_0^x g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \leq C \|g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{t2^k} \log(1+tR)^{\frac{1}{q'}} \|\Delta_{2^k}^x g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^{\frac{1}{2}}(\mathbb{R}_v^D))} \\ &\leq \frac{C}{t2^k} \log(1+tR)^{\frac{1}{q'}} \|g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^{\frac{1}{2}}(\mathbb{R}_v^D))}, \end{aligned}$$

and, if $\beta \neq \frac{1}{2}$,

$$(6.75) \quad \begin{aligned} \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{(t2^k)^{\gamma+\frac{1}{2}}} \|\Delta_{2^k}^x g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \\ &\leq \frac{C}{(t2^k)^{\gamma+\frac{1}{2}}} \|g\phi\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min \left\{ \beta, \frac{1}{2} \right\}$ and $C > 0$ is independent of t and 2^k .

Then, the conclusion of the lemma easily follows from a straightforward application of the methods of paradifferential calculus to the products $f(x, v)\phi(v)$ and $g(x, v)\phi(v)$. For the sake of completeness, we have reproduced in Lemma A.1 the precise estimates that are required here in order to conclude the proof. \square

Proof of Lemma 6.4. For each $\eta \in \mathbb{R}^D$, considering an orthogonal transformation $R_\eta : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that the unit vector $\frac{\eta}{|\eta|}$ is mapped onto $(0, \dots, 0, 1)$ and writing $v' = (v_1, \dots, v_{D-1})$ so that $v = (v', v_D)$, it holds that

$$\begin{aligned} \mathcal{F}_v h_1(\eta, \xi) &= \int_{\mathbb{R}^D} e^{-iv \cdot (R_\eta \xi)} \chi(|v'|) \rho(\lambda v_D) dv \\ (6.76) \quad &= \pi \left((R_\eta \xi)_1, \dots, (R_\eta \xi)_{D-1} \right) \frac{1}{\lambda} \hat{\rho} \left(\frac{1}{\lambda} (R_\eta \xi)_D \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_v h_2(\eta, \xi) &= \mathbb{1}_{\left\{ \frac{R}{2} \leq |\eta| \leq 2R \right\}} \int_{\mathbb{R}^D} e^{-iv \cdot (R_\eta \xi)} \chi(|v'|) \rho(t|\eta|v_D) dv \\ (6.77) \quad &= \mathbb{1}_{\left\{ \frac{R}{2} \leq |\eta| \leq 2R \right\}} \pi \left((R_\eta \xi)_1, \dots, (R_\eta \xi)_{D-1} \right) \frac{1}{t|\eta|} \hat{\rho} \left(\frac{1}{t|\eta|} (R_\eta \xi)_D \right), \end{aligned}$$

where $\pi(\xi') = \int_{\mathbb{R}^{D-1}} e^{-i\xi' \cdot v'} \chi(|v'|) dv' \in \mathcal{S}(\mathbb{R}^{D-1})$.

Consequently, by Plancherel's theorem,

$$\begin{aligned} \|\Delta_0^v h_1\|_{L_v^2} &= \frac{1}{\lambda(2\pi)^{\frac{D}{2}}} \left\| \psi(\xi) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{\lambda} \xi_D \right) \right\|_{L_\xi^2}, \\ (6.78) \quad \|\Delta_{2^k}^v h_1\|_{L_v^2} &= \frac{1}{\lambda(2\pi)^{\frac{D}{2}}} \left\| \varphi \left(\frac{\xi}{2^k} \right) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{\lambda} \xi_D \right) \right\|_{L_\xi^2} \end{aligned}$$

and

$$\begin{aligned} \|\Delta_0^v h_2\|_{L_v^2} &= \mathbb{1}_{\left\{ \frac{R}{2} \leq |\eta| \leq 2R \right\}} \frac{1}{t|\eta|(2\pi)^{\frac{D}{2}}} \left\| \psi(\xi) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{t|\eta|} \xi_D \right) \right\|_{L_\xi^2}, \\ (6.79) \quad \|\Delta_{2^k}^v h_2\|_{L_v^2} &= \mathbb{1}_{\left\{ \frac{R}{2} \leq |\eta| \leq 2R \right\}} \frac{1}{t|\eta|(2\pi)^{\frac{D}{2}}} \left\| \varphi \left(\frac{\xi}{2^k} \right) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{t|\eta|} \xi_D \right) \right\|_{L_\xi^2}. \end{aligned}$$

Recall now that $\text{supp } \psi(\xi) \subset \{|\xi| \leq 1\}$ and $\text{supp } \varphi \left(\frac{\xi}{2^k} \right) \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. In particular, for every $k \in \mathbb{N}$, we have that

$$\begin{aligned} \varphi \left(\frac{\xi}{2^k} \right) &= \varphi \left(\frac{\xi}{2^k} \right) \left(\mathbb{1}_{\{|\xi'| \geq 2^{k-2}\}} + \mathbb{1}_{\{|\xi'| < 2^{k-2}\}} \right) \\ (6.80) \quad &\leq \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \mathbb{1}_{\{|\xi_D| \leq 2^{k+1}\}} + \mathbb{1}_{\{|\xi'| < 2^{k-2}\}} \mathbb{1}_{\{2^{k-2} \leq |\xi_D| \leq 2^{k+1}\}}. \end{aligned}$$

Hence, we deduce that

$$(6.81) \quad \|\Delta_0^v h_1\|_{L_v^2} \leq \frac{C}{\lambda}, \quad \|\Delta_0^v h_2\|_{L_v^2} \leq \frac{C}{tR},$$

and, utilizing that $\pi(\xi')$ decays faster than any polynomial and assuming without any loss of generality that $\hat{\rho}(r)$ is supported inside $\{|r| \leq 2\}$, we further obtain that

$$\begin{aligned}
 \|\Delta_{2^k}^v h_1\|_{L_v^2} &\leq \frac{1}{\lambda} \left\| \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \pi(\xi') \right\|_{L_{\xi'}^2} \left\| \mathbb{1}_{\{|\xi_D| \leq 2^{k+1}\}} \hat{\rho}\left(\frac{1}{\lambda} \xi_D\right) \right\|_{L_{\xi_D}^2} \\
 &\quad + \frac{1}{\lambda} \left\| \mathbb{1}_{\{|\xi'| < 2^{k-2}\}} \pi(\xi') \right\|_{L_{\xi'}^2} \left\| \mathbb{1}_{\{2^{k-2} \leq |\xi_D| \leq 2^{k+1}\}} \hat{\rho}\left(\frac{1}{\lambda} \xi_D\right) \right\|_{L_{\xi_D}^2} \\
 (6.82) \quad &\leq \frac{2^{\frac{k}{2}}}{\lambda} \left\| \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \pi(\xi') \right\|_{L_{\xi'}^2} \left\| \mathbb{1}_{\{|\xi_D| \leq 2\}} \hat{\rho}\left(\frac{2^k}{\lambda} \xi_D\right) \right\|_{L_{\xi_D}^2} \\
 &\quad + C \frac{2^{\frac{k}{2}}}{\lambda} \left\| \mathbb{1}_{\{\frac{1}{4} \leq |\xi_D| \leq 2\}} \hat{\rho}\left(\frac{2^k}{\lambda} \xi_D\right) \right\|_{L_{\xi_D}^2} \\
 &\leq C \left(\frac{2^{\frac{k}{2}}}{\lambda (1 + 2^{k(1-\alpha)})} + \frac{2^{\frac{k}{2}}}{\lambda} \mathbb{1}_{\{\frac{2^k}{\lambda} \leq 8\}} \right)
 \end{aligned}$$

and, similarly, that

$$\begin{aligned}
 \|\Delta_{2^k}^v h_2\|_{L_v^2} &\leq C \mathbb{1}_{\{\frac{R}{2} \leq |\eta| \leq 2R\}} \left(\frac{2^{\frac{k}{2}}}{t|\eta| (1 + 2^{k(1-\alpha)})} + \frac{2^{\frac{k}{2}}}{t|\eta|} \mathbb{1}_{\{\frac{2^k}{t|\eta|} \leq 8\}} \right) \\
 (6.83) \quad &\leq C \left(\frac{2^{\frac{k}{2}}}{tR (1 + 2^{k(1-\alpha)})} + \frac{2^{\frac{k}{2}}}{tR} \mathbb{1}_{\{\frac{2^k}{tR} \leq 16\}} \right).
 \end{aligned}$$

Then, recalling the geometric sum formula $\sum_{k=0}^n x^k = \frac{x^{n+1}-1}{x-1}$, valid for any $x \neq 1$, it follows that, in the case $1 \leq q < \infty$ and $\alpha \neq \frac{1}{2}$,

$$\begin{aligned}
 &\left\| \left\{ 2^{-k\alpha} \|\Delta_{2^k}^v h_1\|_{L_v^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
 (6.84) \quad &\leq \frac{C}{\lambda} \left(\left\| \left\{ \frac{2^{k(\frac{1}{2}-\alpha)}}{1 + 2^{k(1-\alpha)}} \right\}_{k=0}^\infty \right\|_{\ell^q} + \left\| \left\{ 2^{k(\frac{1}{2}-\alpha)} \mathbb{1}_{\{k \leq \frac{\log(8\lambda)}{\log 2}\}} \right\}_{k=0}^\infty \right\|_{\ell^q} \right) \\
 &\leq \frac{C}{\lambda} \left(1 + \left(\frac{(16\lambda)^{(\frac{1}{2}-\alpha)q} - 1}{2^{(\frac{1}{2}-\alpha)q} - 1} \right)^{\frac{1}{q}} \right) \leq \frac{C}{\lambda^{\gamma+\frac{1}{2}}},
 \end{aligned}$$

where $\gamma = \min\{\alpha, \frac{1}{2}\}$, and, similarly, that

$$(6.85) \quad \left\| \left\{ 2^{-k\alpha} \|\Delta_{2^k}^v h_2\|_{L_v^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \leq \frac{C}{(tR)^{\gamma+\frac{1}{2}}}.$$

Furthermore, both cases $q = \infty$ and $\alpha = \frac{1}{2}$ follow from obvious modifications of the preceding estimates. Thus, combining (6.81), (6.84) and (6.85) readily shows that the estimates (6.65) and (6.66) hold.

It only remains to handle the case $\alpha \geq \frac{1}{2}$ and $0 \notin \text{supp} \hat{\rho}$. Without any loss of generality, we may assume that $\hat{\rho}(r)$ is supported inside $\{1 \leq |r| \leq 2\}$. In this

setting, it is possible to refine estimates (6.82) and (6.83) as to obtain

$$\begin{aligned}
\|\Delta_{2^k}^v h_1\|_{L_v^2} &\leq \frac{2^{\frac{k}{2}}}{\lambda} \left\| \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \pi(\xi') \right\|_{L_{\xi'}^2} \left\| \mathbb{1}_{\{|\xi_D| \leq 2\}} \hat{\rho} \left(\frac{2^k}{\lambda} \xi_D \right) \right\|_{L_{\xi_D}^2} \\
&\quad + C \frac{2^{\frac{k}{2}}}{\lambda} \left\| \mathbb{1}_{\{\frac{1}{4} \leq |\xi_D| \leq 2\}} \hat{\rho} \left(\frac{2^k}{\lambda} \xi_D \right) \right\|_{L_{\xi_D}^2} \\
(6.86) \quad &\leq C \left(\frac{2^{\frac{k}{2}}}{\lambda (1+2^k)} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{\lambda}\}} + \frac{2^{\frac{k}{2}}}{\lambda} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{\lambda} \leq 8\}} \right) \\
&\leq C \left(\frac{2^{k\alpha}}{\lambda^{\alpha+\frac{1}{2}} (1+2^k)} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{\lambda}\}} + \frac{2^{k\alpha}}{\lambda^{\alpha+\frac{1}{2}}} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{\lambda} \leq 8\}} \right)
\end{aligned}$$

and, similarly, that

$$\begin{aligned}
\|\Delta_{2^k}^v h_2\|_{L_v^2} &\leq C \mathbb{1}_{\{\frac{R}{2} \leq |\eta| \leq 2R\}} \left(\frac{2^{\frac{k}{2}}}{t|\eta| (1+2^{k(1+\alpha)})} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{t|\eta|}\}} + \frac{2^{\frac{k}{2}}}{t|\eta|} \mathbb{1}_{\{\frac{1}{2} \leq \frac{2^k}{t|\eta|} \leq 8\}} \right) \\
&\leq C \left(\frac{2^{\frac{k}{2}}}{tR (1+2^k)} \mathbb{1}_{\{\frac{1}{4} \leq \frac{2^k}{tR}\}} + \frac{2^{\frac{k}{2}}}{tR} \mathbb{1}_{\{\frac{1}{4} \leq \frac{2^k}{tR} \leq 16\}} \right) \\
&\leq C \left(\frac{2^{k\alpha}}{(tR)^{\alpha+\frac{1}{2}} (1+2^k)} \mathbb{1}_{\{\frac{1}{4} \leq \frac{2^k}{tR}\}} + \frac{2^{k\alpha}}{(tR)^{\alpha+\frac{1}{2}}} \mathbb{1}_{\{\frac{1}{4} \leq \frac{2^k}{tR} \leq 16\}} \right).
\end{aligned}$$

It is then readily seen that, for any $1 \leq q \leq \infty$,

$$\begin{aligned}
&\left\| \left\{ 2^{-k\alpha} \|\Delta_{2^k}^v h_1\|_{L_v^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
(6.88) \quad &\leq \frac{C}{\lambda^{\alpha+\frac{1}{2}}} \left(\left\| \left\{ \frac{1}{1+2^k} \right\}_{k=0}^\infty \right\|_{\ell^q} + \left\| \left\{ \mathbb{1}_{\{\frac{\log \lambda}{\log 2} - 1 \leq k \leq \frac{\log \lambda}{\log 2} + 3\}} \right\}_{k=0}^\infty \right\|_{\ell^q} \right) \\
&\leq \frac{C}{\lambda^{\alpha+\frac{1}{2}}},
\end{aligned}$$

and, similarly, that

$$(6.89) \quad \left\| \left\{ 2^{-k\alpha} \|\Delta_{2^k}^v h_2\|_{L_v^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \leq \frac{C}{(tR)^{\alpha+\frac{1}{2}}}.$$

Finally, notice that, since $\text{supp} \psi(\xi) \subset \{|\xi_D| \leq 1\}$ and $\lambda \geq 1, tR \geq 1$,

$$\begin{aligned}
\|\Delta_0^v h_1\|_{L_v^2} &= \frac{1}{\lambda} \left\| \psi(\xi) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{\lambda} \xi_D \right) \right\|_{L_\xi^2} \mathbb{1}_{\{\lambda < 1\}} = 0, \\
\|\Delta_0^v h_2\|_{L_v^2} &= \mathbb{1}_{\{\frac{R}{2} \leq |\eta| \leq 2R\}} \frac{1}{t|\eta|} \left\| \psi(\xi) \pi(\xi_1, \dots, \xi_{D-1}) \hat{\rho} \left(\frac{1}{t|\eta|} \xi_D \right) \right\|_{L_\xi^2} \mathbb{1}_{\{tR < 1\}} = 0,
\end{aligned}$$

which readily establishes (6.67) and thus, concludes the proof of the lemma. \square

Proof of Lemma 6.5. For each $\eta \in \mathbb{R}^D$, considering an orthogonal transformation $R_\eta : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that $\frac{\eta}{|\eta|}$ is mapped onto $(0, \dots, 0, 1)$ and writing $v' = (v_1, \dots, v_{D-1})$

so that $v = (v', v_D)$, it holds that

(6.91)

$$\begin{aligned} \mathcal{F}_v h(\eta, \xi) &= \mathbb{1}_{\{|\eta| \leq 1\}} \int_{\mathbb{R}^D} e^{-iv \cdot (R_\eta \xi)} \chi(|v|) \rho(|\eta| v_D) dv \\ &= \mathbb{1}_{\{|\eta| \leq 1\}} \int_{\mathbb{R}} \pi \left((R_\eta \xi)_1, \dots, (R_\eta \xi)_{D-1}, (R_\eta \xi)_D - r \right) \frac{1}{|\eta|} \hat{\rho} \left(\frac{1}{|\eta|} r \right) dr, \end{aligned}$$

where $\pi(\xi) = \int_{\mathbb{R}^{D-1}} e^{-i\xi \cdot v} \chi(|v|) dv \in \mathcal{S}(\mathbb{R}^D)$. Consequently,

$$\begin{aligned} \|\Delta_0^v h\|_{L_v^2} &= \mathbb{1}_{\{|\eta| \leq 1\}} \frac{1}{(2\pi)^{\frac{D}{2}}} \left\| \psi(\xi) \int_{\mathbb{R}} \pi(\xi', \xi_D - r) \frac{1}{|\eta|} \hat{\rho} \left(\frac{1}{|\eta|} r \right) dr \right\|_{L_\xi^2} \\ (6.92) \quad &\leq \|\pi(\xi)\|_{L_\xi^2} \|\hat{\rho}(\xi_D)\|_{L_{\xi_D}^1}, \\ \|\Delta_{2^k}^v h\|_{L_v^2} &= \mathbb{1}_{\{|\eta| \leq 1\}} \frac{1}{(2\pi)^{\frac{D}{2}}} \left\| \varphi \left(\frac{\xi}{2^k} \right) \int_{\mathbb{R}} \pi(\xi', \xi_D - r) \frac{1}{|\eta|} \hat{\rho} \left(\frac{1}{|\eta|} r \right) dr \right\|_{L_\xi^2}. \end{aligned}$$

Recall now that $\text{supp} \varphi \left(\frac{\xi}{2^k} \right) \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. In particular, for every $k \in \mathbb{N}$, we have that

$$\begin{aligned} (6.93) \quad \varphi \left(\frac{\xi}{2^k} \right) &= \varphi \left(\frac{\xi}{2^k} \right) \left(\mathbb{1}_{\{|\xi'| \geq 2^{k-2}\}} + \mathbb{1}_{\{|\xi'| < 2^{k-2}\}} \right) \\ &\leq \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \mathbb{1}_{\{|\xi_D| \leq 2^{k+1}\}} + \mathbb{1}_{\{|\xi'| < 2^{k-2}\}} \mathbb{1}_{\{2^{k-2} \leq |\xi_D| \leq 2^{k+1}\}}. \end{aligned}$$

Hence, we deduce, utilizing that $\pi(\xi)$ decays faster than any polynomial and assuming without any loss of generality that $\hat{\rho}(r)$ is supported inside $\{|r| \leq 2\}$, that

$$\begin{aligned} \|\Delta_{2^k}^v h\|_{L_v^2} &\leq \left\| \mathbb{1}_{\{2^{k-2} \leq |\xi'| \leq 2^{k+1}\}} \pi(\xi) \right\|_{L_\xi^2} \|\hat{\rho}(\xi_D)\|_{L_{\xi_D}^1} \\ (6.94) \quad &+ \left\| \mathbb{1}_{\{2^{k-2}-2 \leq |\xi_D| \leq 2^{k+1}+2\}} \pi(\xi) \right\|_{L_\xi^2} \|\hat{\rho}(\xi_D)\|_{L_{\xi_D}^1} \\ &\leq \frac{C}{2^{k(1-\alpha)}}. \end{aligned}$$

It clearly follows that

$$(6.95) \quad \left\| \left\{ 2^{-k\alpha} \|\Delta_{2^k}^v h\|_{L_\eta^\infty L_v^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \leq 2C,$$

which concludes the proof of the lemma. \square

Now that the technical Lemmas 6.3, 6.4 and 6.5 are established, we may proceed to the proof of the main result Theorem 3.5.

Proof of Theorem 3.5. First of all, we easily obtain that

$$\begin{aligned} (6.96) \quad \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} &\leq C \|\Delta_0^x f(x, v)\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \|\phi(v)\|_{B_{2,q'}^{-\alpha}(\mathbb{R}_v^D)} \\ &\leq C \|f(x, v)\|_{B_{2,2,q}^{\alpha,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \|\phi(v)\|_{B_{2,q'}^{-\alpha}(\mathbb{R}_v^D)}, \end{aligned}$$

which concludes the control of the low frequencies. Notice, however, that it will be convenient, in order to carry out interpolation arguments later on, to also estimate

the low frequencies with the same decomposition that was used in the proof of Theorem 3.2 and provided by Proposition 4.1, i.e.

$$(6.97) \quad \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_0^1(f\phi)(x) + B_0^1(g\phi)(x).$$

This is easily done with an application of Lemma 6.3, thus yielding

$$(6.98) \quad \left\| A_0^1(f\phi) \right\|_{L_x^2} \leq C \left\| \Delta_0^x f \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \leq C \|f\|_{B_{2,2,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)},$$

and

$$(6.99) \quad \left\| B_0^1(g\phi) \right\|_{L_x^2} \leq C \left\| \Delta_0^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \leq C \|g\|_{B_{2,2,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}.$$

In order to estimate the high frequencies, according to proposition 4.1, we consider $\rho \in \mathcal{S}(\mathbb{R})$ a cutoff function such that $\tilde{\rho}(r) = \frac{1}{2\pi} \hat{\rho}(-r)$ is compactly supported $\text{supp } \tilde{\rho}(r) \subset \{1 \leq |r| \leq 2\}$ and $\rho(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(r) dr = 1$, so that, for any $t > 0$, we have the dyadic frequency decompositions, for each $k \in \mathbb{N}$,

$$(6.100) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_{2^k}^t(f\phi)(x) + t B_{2^k}^t(g\phi)(x).$$

Consequently, in virtue of lemma 6.3, we infer that, if $\beta \neq \frac{1}{2}$ or $q = 1$, for any $t \geq 2^{-k}$,

$$(6.101) \quad \begin{aligned} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} &\leq \|A_{2^k}^t(f\phi)\|_{L_x^2} + t \|B_{2^k}^t(g\phi)\|_{L_x^2} \\ &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|\Delta_{2^k}^x f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad + 2^{-k} \frac{C}{(t2^k)^{\gamma-\frac{1}{2}}} \|\Delta_{2^k}^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min\{\beta, \frac{1}{2}\}$, and similarly, if $\beta = \frac{1}{2}$ and $q \neq 1$,

$$(6.102) \quad \begin{aligned} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} &\leq \|A_{2^k}^t(f\phi)\|_{L_x^2} + t \|B_{2^k}^t(g\phi)\|_{L_x^2} \\ &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|\Delta_{2^k}^x f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad + \frac{C}{2^k} \log(1 + t2^k)^{\frac{1}{q'}} \|\Delta_{2^k}^x g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of t and 2^k .

Then, optimizing the interpolation parameter t , we choose $t2^k = 2^{k \frac{1+b-a}{1+\alpha-\gamma}}$, which is admissible since $\frac{1+b-a}{1+\alpha-\gamma} \geq 0$ and thus $t \geq 2^{-k}$.

Furthermore, in the cases $\beta > \frac{1}{2}$ or $\beta = \frac{1}{2}$ and $q = 1$, we can choose $t = \infty$, which is, in fact, more optimal than $t2^k = 2^{k \frac{1+b-a}{1+\alpha-\gamma}}$, since it eliminates the first term in the right-hand side of the above estimates. This case is discussed later on.

It follows that, recalling $s = (1 + b - a) \frac{\alpha + \frac{1}{2}}{1 + \alpha - \gamma} + a = (1 + b - a) \frac{\gamma - \frac{1}{2}}{1 + \alpha - \gamma} + 1 + b$, in the case $\beta \neq \frac{1}{2}$ or $q = 1$,

$$(6.103) \quad \begin{aligned} & 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \\ & \leq C 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} + C 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

and similarly, if $\beta = \frac{1}{2}$ and $q \neq 1$, for every $\epsilon > 0$,

$$(6.104) \quad \begin{aligned} & 2^{k(s-\epsilon)} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \\ & \leq C 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} + C 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Finally, summing over $k \in \mathbb{N}$ yields that, in the case $\beta \neq \frac{1}{2}$ or $q = 1$,

$$(6.105) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \quad + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=0}^\infty \right\|_{\ell^q}, \end{aligned}$$

and similarly, when $\beta = \frac{1}{2}$ and $q \neq 1$,

$$(6.106) \quad \begin{aligned} & \left\| \left\{ 2^{k(s-\epsilon)} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \quad + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=0}^\infty \right\|_{\ell^q}. \end{aligned}$$

We handle now the cases $\beta > \frac{1}{2}$ or $\beta = \frac{1}{2}$ and $q = 1$, by letting t tend to infinity in (6.101), as mentioned previously. This leads to

$$(6.107) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \leq \frac{C}{2^k} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))}.$$

Hence, recalling $s = 1 + b$ and summing over k yields

$$(6.108) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^2} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=0}^\infty \right\|_{\ell^q}, \end{aligned}$$

which concludes the proof of the theorem. \square

6.4. The $L_x^1 L_v^p$ and $L_x^2 L_v^2$ cases reconciled. We give now the proof of the general Theorem 3.6, which will follow from a simple interpolation procedure. In particular, the following lemma results from the interpolation between Lemma 6.1 and Lemma 6.3.

Lemma 6.6. *Let $\phi(v) \in C_0^\infty(\mathbb{R}^D)$. For every $1 \leq p, r \leq \infty$, $1 \leq q < \infty$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $t \geq 2^{-k}$ such that*

$$(6.109) \quad r \leq p \leq r',$$

it holds that

$$(6.110) \quad \|A_0^1(f\phi)\|_{L_x^p} \leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))}$$

and

$$(6.111) \quad \|A_{2^k}^t(f\phi)\|_{L_x^p} \leq \frac{C}{t^{\alpha+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})}} 2^{k(\alpha+1-\frac{1}{r})} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))},$$

where $C > 0$ is independent of t and 2^k .

Furthermore, if

$$(6.112) \quad D\left(\frac{1}{r} - \frac{1}{p}\right) < \frac{2}{r} - 1 \quad \text{or} \quad p = r = 2,$$

then, for any $\beta \in \mathbb{R}$, it holds that

$$(6.113) \quad \|B_0^1(g\phi)\|_{L_x^p} \leq C \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

and, if further $\beta \neq \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$,

$$(6.114) \quad \|B_{2^k}^t(g\phi)\|_{L_x^p} \leq \frac{C}{t^{\gamma+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})}} 2^{k(\gamma+1-\frac{1}{r})} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

where $\gamma = \min\left\{\beta, \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)\right\}$ and $C > 0$ is independent of t and 2^k .

Finally, if $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$, then

$$(6.115) \quad \|B_{2^k}^t(g\phi)\|_{L_x^p} \leq \frac{C}{t^{2k(\beta+1-\frac{1}{r})}} \log(1+t2^k)^{\frac{1}{q'}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

where $C > 0$ is independent of t and 2^k .

Proof. First, if $r = 1$, then the statement of the present lemma is a straightforward consequence of Lemma 6.1 and Lemma A.1, while the case $r = 2$ is contained in Lemma 6.3. If $1 < r < 2$, then the result will follow from the interpolation of Lemma 6.1 with Lemma 6.3. To this end, we define the interpolation parameter $0 < \lambda < 1$ by $\lambda = 2\left(1 - \frac{1}{r}\right)$, so that

$$(6.116) \quad \frac{1}{r} = \frac{1-\lambda}{1} + \frac{\lambda}{2},$$

and the parameter $1 \leq p_0 < \infty$ by

$$(6.117) \quad \frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{2}.$$

It is then easy to check that if $D \left(\frac{1}{r} - \frac{1}{p} \right) < \frac{2}{r} - 1$, then

$$(6.118) \quad D \left(1 - \frac{1}{p_0} \right) < 1.$$

Further notice that

$$(6.119) \quad \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right) = (1 - \lambda) \left[1 - D \left(1 - \frac{1}{p_0} \right) \right] + \frac{\lambda}{2}.$$

Therefore, if $\beta \left\{ \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \right\} \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$, it is always possible to respectively find $\beta_0 \left\{ \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \right\} 1 - D \left(1 - \frac{1}{p_0} \right)$ and $\beta_1 \left\{ \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \right\} \frac{1}{2}$ such that

$$(6.120) \quad \beta = (1 - \lambda)\beta_0 + \lambda\beta_1.$$

Consequently, defining $\gamma_0 = \min \left\{ \beta_0, 1 - D \left(1 - \frac{1}{p_0} \right) \right\}$ and $\gamma_1 = \min \left\{ \beta_1, \frac{1}{2} \right\}$, it also holds that

$$(6.121) \quad \gamma = (1 - \lambda)\gamma_0 + \lambda\gamma_1.$$

Then, in the case $\beta \neq \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$, we deduce according to Lemma 6.1 that

$$(6.122) \quad \begin{aligned} \|A_0^1(f\phi)\|_{L_x^{p_0}} &\leq C \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p_0,q}^\alpha(\mathbb{R}_v^D))}, \\ \|A_{2^k}^t(f\phi)\|_{L_x^{p_0}} &\leq \frac{C}{t^{\alpha+D(1-\frac{1}{p_0})} 2^{k\alpha}} \|f\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p_0,q}^\alpha(\mathbb{R}_v^D))}, \\ \|B_0^1(g\phi)\|_{L_x^{p_0}} &\leq C \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p_0,q}^{\beta_0}(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^{p_0}} &\leq \frac{C}{t^{\gamma_0+D(1-\frac{1}{p_0})} 2^{k\gamma_0}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p_0,q}^{\beta_0}(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of t and 2^k , and employing Lemma 6.3, we infer that

$$(6.123) \quad \begin{aligned} \|A_0^1(f\phi)\|_{L_x^2} &\leq C \|f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))}, \\ \|A_{2^k}^t(f\phi)\|_{L_x^2} &\leq \frac{C}{(t2^k)^{\alpha+\frac{1}{2}}} \|f\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^\alpha(\mathbb{R}_v^D))}, \\ \|B_0^1(g\phi)\|_{L_x^2} &\leq C \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^{\beta_1}(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{(t2^k)^{\gamma_1+\frac{1}{2}}} \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^{\beta_1}(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of t and 2^k .

In the case $\beta = \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$, according to the same Lemmas 6.1 and 6.3, solely the estimates on $B_{2^k}^t$ should be replaced by

$$(6.124) \quad \begin{aligned} \|B_{2^k}^t(g\phi)\|_{L_x^{p_0}} &\leq \frac{C}{t2^{k\beta_0}} \log \left(1 + t2^k \right)^{\frac{1}{q}} \|g\|_{\tilde{L}^1(\mathbb{R}_x^D; B_{p_0,q}^{\beta_0}(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^2} &\leq \frac{C}{t2^k} \log \left(1 + t2^k \right)^{\frac{1}{q}} \|g\|_{\tilde{L}^2(\mathbb{R}_x^D; B_{2,q}^{\beta_1}(\mathbb{R}_v^D))}. \end{aligned}$$

We are now going to utilize standard results from complex interpolation theory (cf. [6]) in order to obtain new estimates from the interpolation of estimates (6.122) and (6.123).

To this end, first recall that the real interpolation of Lebesgue spaces (cf. [6]) yields that $(L^{p_0}, L^{p_1})_{[\theta]} = L^p$, for any $1 \leq p, p_0, p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Furthermore, the complex interpolation of standard Besov spaces (cf. [6]) yields, in particular, that $(B_{p_0,q}^{\alpha_0}, B_{p_1,q}^{\alpha_1})_{[\theta]} = B_{p,q}^{\alpha}$, for any $\alpha, \alpha_0, \alpha_1 \in \mathbb{R}$ and $1 \leq p, p_0, p_1, q < \infty$ such that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. It is then possible to smoothly adapt the proof of this standard property to obtain a corresponding result for the Besov spaces introduced in Section 2.1. Namely, one can show that $(\tilde{L}^{r_0} B_{p_0,q}^{\alpha_0}, \tilde{L}^{r_1} B_{p_1,q}^{\alpha_1})_{[\theta]} = \tilde{L}^r B_{p,q}^{\alpha}$, for any $\alpha, \alpha_0, \alpha_1 \in \mathbb{R}$ and $1 \leq p, p_0, p_1, q, r, r_0, r_1 < \infty$ such that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$.

Therefore, we deduce from the complex interpolation of (6.122) and (6.123) that, in the case $\beta \neq \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$,

$$\begin{aligned}
 (6.125) \quad & \|A_0^1(f\phi)\|_{L_x^p} \leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\alpha}(\mathbb{R}_v^D))}, \\
 & \|A_{2^k}^t(f\phi)\|_{L_x^p} \leq \frac{C}{\left(t^{\alpha+D(1-\frac{1}{p_0})} 2^{k\alpha}\right)^{1-\lambda} \left((t2^k)^{\alpha+\frac{1}{2}}\right)^{\lambda}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\alpha}(\mathbb{R}_v^D))} \\
 & = \frac{C}{t^{\alpha+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})} 2^{k(\alpha+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\alpha}(\mathbb{R}_v^D))},
 \end{aligned}$$

and

$$\begin{aligned}
 (6.126) \quad & \|B_0^1(g\phi)\|_{L_x^p} \leq C \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))}, \\
 & \|B_{2^k}^t(g\phi)\|_{L_x^p} \leq \frac{C}{\left(t^{\gamma_0+D(1-\frac{1}{p_0})} 2^{k\gamma_0}\right)^{1-\lambda} \left((t2^k)^{\gamma_1+\frac{1}{2}}\right)^{\lambda}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))} \\
 & = \frac{C}{t^{\gamma+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})} 2^{k(\gamma+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))},
 \end{aligned}$$

where $C > 0$ is independent of t and 2^k .

Accordingly, in the case $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$, only the estimate on $B_{2^k}^t$ should be replaced by

$$\begin{aligned}
 (6.127) \quad & \|B_{2^k}^t(g\phi)\|_{L_x^p} \leq \frac{C}{(t2^{k\beta_0})^{1-\lambda} (t2^k)^{\lambda}} \log(1+t2^k)^{\frac{1}{q'}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))} \\
 & = \frac{C}{t^{2^{k(\beta+1-\frac{1}{r})}} \log(1+t2^k)^{\frac{1}{q'}}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta}(\mathbb{R}_v^D))},
 \end{aligned}$$

where $C > 0$ is independent of t and 2^k , which concludes the proof of the lemma. \square

Proof of Theorem 3.6. As usual, we begin with the decompositions

$$(6.128) \quad \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_0^1(f\phi)(x) + B_0^1(g\phi)(x)$$

and

$$(6.129) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_{2^k}^t(f\phi)(x) + tB_{2^k}^t(g\phi)(x).$$

provided by Proposition 4.1.

We show in Lemma 6.6 that, in the case $\beta \neq \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ or $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and $q = 1$, the operators above satisfy the bounds

$$(6.130) \quad \begin{aligned} \|A_0^1(f\phi)\|_{L_x^p} &\leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))}, \\ \|A_{2^k}^t(f\phi)\|_{L_x^p} &\leq \frac{C}{t^{\alpha+1-\frac{1}{r}+D\left(\frac{1}{r}-\frac{1}{p}\right)} 2^{k(\alpha+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))}, \\ \|B_0^1(g\phi)\|_{L_x^p} &\leq C \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \\ \|B_{2^k}^t(g\phi)\|_{L_x^p} &\leq \frac{C}{t^{\gamma+1-\frac{1}{r}+D\left(\frac{1}{r}-\frac{1}{p}\right)} 2^{k(\gamma+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $\gamma = \min\left\{\beta, \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)\right\}$ and $C > 0$ is independent of t and 2^k .

It then follows that, in virtue of the identities (4.25) from Proposition 4.1,

$$(6.131) \quad \begin{aligned} \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} &\leq \|A_0^1(S_2^x f\phi)\|_{L_x^p} + \|B_0^1(S_2^x g\phi)\|_{L_x^p} \\ &\leq C \|S_2^x f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} + C \|S_2^x g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \\ &\leq C \|f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + C \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}, \end{aligned}$$

which concludes the estimate on the low frequencies.

Regarding the high frequencies, we obtain

$$(6.132) \quad \begin{aligned} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} &\leq \left\| A_{2^k}^t \left(\Delta_{[2^{k-1}, 2^{k+1}]}^x f\phi \right) \right\|_{L_x^p} + t \left\| B_{2^k}^t \left(\Delta_{[2^{k-1}, 2^{k+1}]}^x g\phi \right) \right\|_{L_x^p} \\ &\leq C \frac{1}{t^{\alpha+1-\frac{1}{r}+D\left(\frac{1}{r}-\frac{1}{p}\right)} 2^{k(\alpha+1-\frac{1}{r})}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \\ &\quad + C \frac{t^{\frac{1}{r}-\gamma-D\left(\frac{1}{r}-\frac{1}{p}\right)}}{2^{k(\gamma+1-\frac{1}{r})}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Next, optimizing in t for each value of k , we fix the interpolation parameter t as $t_k = 2^{-k \frac{(\alpha-\gamma)+(a-b)}{1+\alpha-\gamma}}$. Note that $t_k \geq 2^{-k}$, for $b \geq a - 1$, and that this choice is independent of $1 \leq p, q, r \leq \infty$.

Furthermore, in the cases $\beta > \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ or $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and $q = 1$, we can choose $t = \infty$, which is, in fact, more optimal than $t2^k = 2^{k\frac{1+b-a}{1+\alpha-\gamma}}$, since it eliminates the first term in the right-hand side of the above estimates. This case is discussed later on.

Therefore, denoting $s = (1+b-a)\frac{\alpha+1-\frac{1}{r}+D\left(\frac{1}{r}-\frac{1}{p}\right)}{1+\alpha-\gamma} + a - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and setting $t = t_k$, we find that

$$(6.133) \quad \begin{aligned} & 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \\ & \leq C 2^{ka} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} + C 2^{kb} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}. \end{aligned}$$

Hence, summing over k , we obtain

$$(6.134) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\ & + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q} \\ & = C \|f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + C \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}, \end{aligned}$$

which concludes the proof of the theorem in the case $\beta < \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$.

We handle now the cases $\beta > \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ or $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and $q = 1$, by letting t tend to infinity in (6.132), as mentioned previously. This leads to

$$(6.135) \quad \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \leq \frac{C}{2^{k(1-D\left(\frac{1}{r}-\frac{1}{p}\right))}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}.$$

Hence, recalling $s = 1 + b - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and summing over k yields

$$(6.136) \quad \begin{aligned} & \left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\ & \leq C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^q}, \end{aligned}$$

which concludes the proof of the theorem in the cases $\beta > \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ or $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and $q = 1$.

As for the case $\beta = \frac{1}{r} - D\left(\frac{1}{r} - \frac{1}{p}\right)$ and $q \neq 1$, employing the corresponding estimate from Lemma 6.6, we find that, as in the proofs of Theorems 3.2 and 3.5,

for every $\epsilon > 0$,

$$(6.137) \quad \left\| \left\{ 2^{k(s-\epsilon)} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \leq C \|f\|_{B_{r,p,q}^{a,\alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} + C \|g\|_{B_{r,p,q}^{b,\beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)},$$

which concludes the proof of the theorem. \square

We proceed now to the proof of the most general Theorem 3.7, which will follow from a quite involved interpolation procedure. In particular, the following lemma is a generalization of Lemma 6.6.

Lemma 6.7. *Let $\phi(v) \in C_0^\infty(\mathbb{R}^D)$. For every $1 \leq p, q, r \leq \infty$, $0 < m < \infty$, $\alpha, \lambda \in \mathbb{R}$ and $k \in \mathbb{N}$ such that*

$$(6.138) \quad r \leq p \leq r'$$

and

$$(6.139) \quad \frac{1}{m} = \alpha + 1 - \frac{1}{r} + D \left(\frac{1}{r} - \frac{1}{p} \right) - \lambda,$$

it holds that

$$(6.140) \quad \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} t^\lambda \|A_{2^k}^t(f\phi)\|_{L_x^p} \right\|_{L_t^{m,q}} \leq \frac{C}{2^{k(\alpha+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))},$$

where $C > 0$ is independent of 2^k .

Furthermore, if

$$(6.141) \quad D \left(\frac{1}{r} - \frac{1}{p} \right) < \frac{2}{r} - 1 \quad \text{or} \quad p = r = 2,$$

then, for any $0 < n < \infty$ and $\beta, \tau \in \mathbb{R}$ such that

$$(6.142) \quad \beta < \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$$

and

$$(6.143) \quad \frac{1}{n} = \beta + 1 - \frac{1}{r} + D \left(\frac{1}{r} - \frac{1}{p} \right) - \tau,$$

it holds that

$$(6.144) \quad \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} t^\tau \|B_{2^k}^t(g\phi)\|_{L_x^p} \right\|_{L_t^{n,q}} \leq \frac{C}{2^{k(\beta+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))},$$

where $C > 0$ is independent of 2^k .

Proof. We proceed by interpolation of the estimates from Lemma 6.6 on the velocity regularity index, which in particular illustrates the importance of systematically dealing with the most general cases of function spaces as possible.

To this end, consider $\alpha_0 < \alpha < \alpha_1$, $\beta_0 < \beta < \beta_1$, $0 < m_1 < m < m_0 < \infty$ and $0 < n_1 < n < n_0 < \infty$, such that

$$(6.145) \quad \alpha = \frac{\alpha_0 + \alpha_1}{2}, \quad \beta = \frac{\beta_0 + \beta_1}{2}, \quad \beta_1 < \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right),$$

and, for each $i = 0, 1$,

$$(6.146) \quad \begin{aligned} \frac{1}{m_i} &= \alpha_i + 1 - \frac{1}{r} + D \left(\frac{1}{r} - \frac{1}{p} \right) - \lambda, \\ \frac{1}{n_i} &= \beta_i + 1 - \frac{1}{r} + D \left(\frac{1}{r} - \frac{1}{p} \right) - \tau. \end{aligned}$$

Then, using Lemma 6.6, it holds that, for each $i = 0, 1$ and every $t \geq 2^{-k}$,

$$(6.147) \quad \|A_{2^k}^t(f\phi)\|_{L_x^p} \leq \frac{C}{t^{\alpha_i+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})} 2^{k(\alpha_i+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\alpha_i}(\mathbb{R}_v^D))}$$

and, if further $D \left(\frac{1}{r} - \frac{1}{p} \right) < \frac{2}{r} - 1$ or $p = r = 2$,

$$(6.148) \quad \|B_{2^k}^t(g\phi)\|_{L_x^p} \leq \frac{C}{t^{\beta_i+1-\frac{1}{r}+D(\frac{1}{r}-\frac{1}{p})} 2^{k(\beta_i+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta_i}(\mathbb{R}_v^D))}.$$

It follows that

$$(6.149) \quad \begin{aligned} \|\mathbb{1}_{\{t \geq 2^{-k}\}} t^\lambda \|A_{2^k}^t(f\phi)\|_{L_x^p}\|_{L_t^{m_i,\infty}} &\leq \frac{C}{2^{k(\alpha_i+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\alpha_i}(\mathbb{R}_v^D))}, \\ \|\mathbb{1}_{\{t \geq 2^{-k}\}} t^\tau \|B_{2^k}^t(g\phi)\|_{L_x^p}\|_{L_t^{n_i,\infty}} &\leq \frac{C}{2^{k(\beta_i+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{\beta_i}(\mathbb{R}_v^D))}, \end{aligned}$$

where $L^{p,q}$ denotes the standard Lorentz spaces.

Recall now that the real interpolation of Lorentz spaces (cf. [6]) yields, in particular, that $(L^{m_0,\infty} A, L^{m_1,\infty} A)_{\frac{1}{2},q} = L^{m,q} A$, where A is any fixed Banach space, for any $0 < m, m_0, m_1, q \leq \infty$ such that $\frac{1}{m} = \frac{1}{2} \left(\frac{1}{m_0} + \frac{1}{m_1} \right)$ and $m_0 \neq m_1$.

Furthermore, the real interpolation of standard Besov spaces (cf. [6]) yields, in particular, that $(B_{p,q}^{\alpha_0}, B_{p,q}^{\alpha_1})_{\frac{1}{2},c} = B_{p,c}^\alpha$, for any $\alpha, \alpha_0, \alpha_1 \in \mathbb{R}$ and $1 \leq p, q, c \leq \infty$ such that $\alpha = \frac{\alpha_0 + \alpha_1}{2}$ and $\alpha_0 \neq \alpha_1$. It is then possible to smoothly adapt the proof of this standard property to obtain a corresponding result for the Besov spaces introduced in Section 2.1. Namely, one can show that $(\tilde{L}^r B_{p,q}^{\alpha_0}, \tilde{L}^r B_{p,q}^{\alpha_1})_{\frac{1}{2},c} = \tilde{L}^r B_{p,c}^\alpha$, for any $\alpha, \alpha_0, \alpha_1 \in \mathbb{R}$ and $1 \leq p, q, c \leq \infty$ such that $\alpha = \frac{\alpha_0 + \alpha_1}{2}$ and $\alpha_0 \neq \alpha_1$.

Therefore, we deduce from (6.149) that

$$(6.150) \quad \begin{aligned} \|\mathbb{1}_{\{t \geq 2^{-k}\}} t^\lambda \|A_{2^k}^t(f\phi)\|_{L_x^p}\|_{L_t^{m,q}} &\leq \frac{C}{2^{k(\alpha+1-\frac{1}{r})}} \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\alpha(\mathbb{R}_v^D))}, \\ \|\mathbb{1}_{\{t \geq 2^{-k}\}} t^\tau \|B_{2^k}^t(g\phi)\|_{L_x^p}\|_{L_t^{n,q}} &\leq \frac{C}{2^{k(\beta+1-\frac{1}{r})}} \|g\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

which concludes the proof of the lemma.

Note that, unfortunately, in the case $\beta \geq \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right)$, we cannot use interpolation methods to improve the results of Lemma 6.6 because $\gamma - \frac{1}{r} + D \left(\frac{1}{r} - \frac{1}{p} \right) = 0$, where $\gamma = \min \left\{ \beta, \frac{1}{r} - D \left(\frac{1}{r} - \frac{1}{p} \right) \right\}$, remains constant over that range of parameters. \square

Proof of Theorem 3.7. As usual, we begin with the decompositions

$$(6.151) \quad \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_0^1(f\phi)(x) + B_0^1(g\phi)(x)$$

and

$$(6.152) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = A_{2^k}^t(f\phi)(x) + tB_{2^k}^t(g\phi)(x).$$

provided by Proposition 4.1.

We have shown in Lemma 6.7 above that the operators $A_{2^k}^t$ and $B_{2^k}^t$ satisfy the bounds

$$(6.153) \quad \begin{aligned} \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} t^\lambda \|A_{2^k}^t(f\phi)\|_{L_x^{p_0}} \right\|_{L_t^{q_0}} &\leq \frac{C}{2^{k(\alpha+1-\frac{1}{r_0})}} \|f\|_{\tilde{L}^{r_0}(\mathbb{R}_x^D; B_{p_0, q_0}^\alpha(\mathbb{R}_v^D))}, \\ \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} t^\tau \|B_{2^k}^t(g\phi)\|_{L_x^{p_1}} \right\|_{L_t^{q_1}} &\leq \frac{C}{2^{k(\beta+1-\frac{1}{r_1})}} \|g\|_{\tilde{L}^{r_1}(\mathbb{R}_x^D; B_{p_1, q_1}^\beta(\mathbb{R}_v^D))}, \end{aligned}$$

where $C > 0$ is independent of 2^k and

$$(6.154) \quad \begin{aligned} \lambda &= \alpha + 1 - \frac{1}{r_0} + D \left(\frac{1}{r_0} - \frac{1}{p_0} \right) - \frac{1}{q_0}, \\ \tau &= \beta + 1 - \frac{1}{r_1} + D \left(\frac{1}{r_1} - \frac{1}{p_1} \right) - \frac{1}{q_1}. \end{aligned}$$

We will now make use of the construction of interpolation spaces known as *espaces de moyennes* presented in section 2.2. Specifically, in virtue of the property $(L^{p_0}, L^{p_1})_{\theta, p} = L^p$ valid for all $1 \leq p, p_0, p_1 \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we wish to express the Lebesgue space L^p using the norm (2.23). That is to say, we will employ the norm

$$(6.155) \quad \inf_{a=a_0+a_1} \left(\left\| s^{-\theta} a_0(s) \right\|_{L^{q_0}((0, \infty), \frac{dx}{s}; L^{p_0}(\mathbb{R}^D, dx))}^{q_0} + \left\| s^{1-\theta} a_1(s) \right\|_{L^{q_1}((0, \infty), \frac{dx}{s}; L^{p_1}(\mathbb{R}^D, dx))}^{q_1} \right)^{\frac{1}{p}},$$

which is equivalent to the usual norm on $L^p(\mathbb{R}^D, dx)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, provided $\frac{1}{p} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $q_0, q_1 < \infty$.

To this end, for some suitable bijective function $t(s) : (0, \infty) \rightarrow (0, \infty)$ to be determined later on, we will decompose

$$(6.156) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = a = a_0(s) + a_1(s) = \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv + 0,$$

when $0 < t(s) < 2^{-k}$, and

$$(6.157) \quad \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv = a = a_0(s) + a_1(s) = A_{2^k}^t(f\phi)(x) + tB_{2^k}^t(g\phi)(x),$$

when $t(s) \geq 2^{-k}$. It follows that, using Bernstein's inequalities,

$$\begin{aligned}
 (6.158) \quad & \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \\
 & \leq C \left\| \mathbb{1}_{\{0 < t < 2^{-k}\}} s^{-\theta - \frac{1}{q_0}} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^{p_0}} \right\|_{L_s^{q_0}}^{\frac{q_0}{p}} \\
 & + C \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} s^{-\theta - \frac{1}{q_0}} \|A_{2^k}^t(f\phi)\|_{L_x^{p_0}} \right\|_{L_s^{q_0}}^{\frac{q_0}{p}} + C \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} s^{1-\theta - \frac{1}{q_1}} t \|B_{2^k}^t(g\phi)\|_{L_x^{p_1}} \right\|_{L_s^{q_1}}^{\frac{q_1}{p}} \\
 & \leq C 2^{kD(\frac{1}{r_0} - \frac{1}{p_0}) \frac{q_0}{p}} \left\| \mathbb{1}_{\{0 < t < 2^{-k}\}} s(t)^{-\theta} \left(\frac{s'(t)}{s(t)} \right)^{\frac{1}{q_0}} \right\|_{L_t^{q_0}}^{\frac{q_0}{p}} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^{p_0}}^{\frac{q_0}{p}} \\
 & + C \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} s(t)^{-\theta} \left(\frac{s'(t)}{s(t)} \right)^{\frac{1}{q_0}} \|A_{2^k}^t(f\phi)\|_{L_x^{p_0}} \right\|_{L_t^{q_0}}^{\frac{q_0}{p}} \\
 & + C \left\| \mathbb{1}_{\{t \geq 2^{-k}\}} s(t)^{1-\theta} \left(\frac{s'(t)}{s(t)} \right)^{\frac{1}{q_1}} t \|B_{2^k}^t(g\phi)\|_{L_x^{p_1}} \right\|_{L_t^{q_1}}^{\frac{q_1}{p}}.
 \end{aligned}$$

Next, we set the dependence of s with respect to t so that we may utilize the estimates (6.153). This amounts to optimizing the value of the interpolation parameter s for each value of t , which also dictates the value of $0 < \theta < 1$. That is to say, for given coefficients $c_1, c_2 > 0$ independent of k , we wish to find an optimal value for a function of the form

$$(6.159) \quad c_1 \frac{s(t)^{-\theta}}{2^{k(\alpha+1-\frac{1}{r_0})} t^\lambda} \left(\frac{s'(t)}{s(t)} \right)^{\frac{1}{q_0}} + c_2 \frac{s(t)^{1-\theta}}{2^{k(\beta+1-\frac{1}{r_1})} t^\tau} \left(\frac{s'(t)}{s(t)} \right)^{\frac{1}{q_1}} t.$$

Thus, we define

$$\begin{aligned}
 (6.160) \quad s(t) &= \frac{2^{k(\beta+b+1-\frac{1}{r_1})} t^{\tau+\frac{1}{q_1}-1}}{2^{k(\alpha+a+1-\frac{1}{r_0})} t^{\lambda+\frac{1}{q_0}}} \\
 &= 2^{k((\beta+b)-(\alpha+a)+\frac{1}{r_0}-\frac{1}{r_1})} t^{\beta-\alpha-1+\frac{1}{r_0}-\frac{1}{r_1}+D(\frac{1}{r_1}-\frac{1}{r_0}-\frac{1}{p_1}+\frac{1}{p_0})},
 \end{aligned}$$

which is admissible since $\beta - \alpha - 1 + \frac{1}{r_0} - \frac{1}{r_1} + D\left(\frac{1}{r_1} - \frac{1}{r_0} - \frac{1}{p_1} + \frac{1}{p_0}\right) < 0$. Furthermore, in order that the resulting terms $\frac{s(t)^{-\theta}}{t^{\lambda+\frac{1}{q_0}}}$ and $\frac{s(t)^{1-\theta}}{t^{\tau+\frac{1}{q_1}-1}}$ be independent of t , we set $0 < \theta < 1$ so that $(1-\theta)\left(\lambda + \frac{1}{q_0}\right) + \theta\left(\tau + \frac{1}{q_1} - 1\right) = 0$, which results in

$$(6.161) \quad \theta = \frac{\alpha + 1 - \frac{1}{r_0} + D\left(\frac{1}{r_0} - \frac{1}{p_0}\right)}{\alpha + 1 - \frac{1}{r_0} + D\left(\frac{1}{r_0} - \frac{1}{p_0}\right) - \beta + \frac{1}{r_1} - D\left(\frac{1}{r_1} - \frac{1}{p_1}\right)}.$$

To be precise, these choices of parameters yield that

$$(6.162) \quad \begin{aligned} s(t)^{-\theta} &= 2^{-ks} 2^{k(a+\alpha+1-\frac{1}{r_0})} t^{\lambda+\frac{1}{q_0}}, \\ s(t)^{1-\theta} &= 2^{-ks} 2^{k(b+\beta+1-\frac{1}{r_1})} t^{\tau-1+\frac{1}{q_1}}, \end{aligned}$$

where $s = (1-\theta) \left(\alpha + a + 1 - \frac{1}{r_0} \right) + \theta \left(\beta + b + 1 - \frac{1}{r_1} \right)$.

We conclude that, in virtue of the identities (4.25) from Proposition 4.1,

$$(6.163) \quad \begin{aligned} \left(2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right)^p &\leq C \left(2^{ka} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x f \right\|_{\tilde{L}^{r_0}(\mathbb{R}_x^D; B_{p_0, q_0}^\alpha(\mathbb{R}_v^D))} \right)^{q_0} \\ &\quad + C \left(2^{kb} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^x g \right\|_{\tilde{L}^{r_1}(\mathbb{R}_x^D; B_{p_1, q_1}^\beta(\mathbb{R}_v^D))} \right)^{q_1}. \end{aligned}$$

Hence, summing over k , we obtain

$$(6.164) \quad \begin{aligned} &\left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^p} \\ &\leq C \left\| \left\{ 2^{ka} \left\| \Delta_{2^k}^x f \right\|_{\tilde{L}^{r_0}(\mathbb{R}_x^D; B_{p_0, q_0}^\alpha(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^{q_0}}^{\frac{q_0}{p}} \\ &\quad + C \left\| \left\{ 2^{kb} \left\| \Delta_{2^k}^x g \right\|_{\tilde{L}^{r_1}(\mathbb{R}_x^D; B_{p_1, q_1}^\beta(\mathbb{R}_v^D))} \right\}_{k=-1}^\infty \right\|_{\ell^{q_1}}^{\frac{q_1}{p}} \\ &\leq C \|f\|_{B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{q_0}{p}} + C \|g\|_{B_{r_1, p_1, q_1}^{b, \beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{q_1}{p}}. \end{aligned}$$

Now, if the above estimate holds true for any f and g , then it must also be valid for all λf and λg , where $\lambda > 0$, so that

$$(6.165) \quad \begin{aligned} &\left\| \left\{ 2^{ks} \left\| \Delta_{2^k}^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \right\}_{k=0}^\infty \right\|_{\ell^p} \\ &\leq C \lambda^{\frac{q_0}{p}-1} \|f\|_{B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{q_0}{p}} + C \lambda^{\frac{q_1}{p}-1} \|g\|_{B_{r_1, p_1, q_1}^{b, \beta}(\mathbb{R}_x^D \times \mathbb{R}_v^D)}^{\frac{q_1}{p}}. \end{aligned}$$

Recalling that $\frac{1}{p} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and optimizing in λ concludes the proof of the main estimate for the high frequencies of the velocity average.

Thus, there only remains to control the low frequencies $\Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv$. To this end, if $p \geq r_0$, a straightforward application of Bernstein's inequalities

shows that

(6.166)

$$\begin{aligned}
& \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \phi(v) dv \right\|_{L_x^p} \\
& \leq C \left\| \int_{\mathbb{R}^D} \Delta_0^x f(x, v) \phi(v) dv \right\|_{L_x^{r_0}} \\
& = C \left\| \int_{\mathbb{R}^D} \Delta_0^x \Delta_0^v f(x, v) S_2^v \phi(v) dv + \sum_{k=0}^{\infty} \int_{\mathbb{R}^D} \Delta_0^x \Delta_{2^k}^v f(x, v) \Delta_{[2^{k-1}, 2^{k+1}]}^v \phi(v) dv \right\|_{L_x^{r_0}} \\
& \leq C \left\| \Delta_0^x \Delta_0^v f \right\|_{L_x^{r_0} L_v^{p_0}} \left\| S_2^v \phi \right\|_{L_v^{p'_0}} + C \sum_{k=0}^{\infty} \left\| \Delta_0^x \Delta_{2^k}^v f \right\|_{L_x^{r_0} L_v^{p_0}} \left\| \Delta_{[2^{k-1}, 2^{k+1}]}^v \phi \right\|_{L_v^{p'_0}} \\
& \leq C \|f\|_{B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \|\phi\|_{B_{p'_0, q'_0}^{-\alpha}(\mathbb{R}_v^D)}.
\end{aligned}$$

Finally, in the case $p < r_0$, the direct use of Bernstein's inequalities as above is not allowed and it is therefore necessary to localize the norm in space in order to carry out the preceding argument. Thus, for any $\chi(x) \in C_0^\infty(\mathbb{R}^D)$, we employ the standard methods of paradifferential calculus, which are used in the proof of Lemma A.1, to decompose, following Bony's method,

(6.167)

$$\Delta_0^x (f(x, v) \chi(x)) = \Delta_0^x \left[\Delta_0^x f S_4^x \chi + \Delta_1^x f S_8^x \chi + \Delta_2^x f S_{16}^x \chi + \sum_{k=2}^{\infty} \Delta_{2^k}^x f \Delta_{[2^{k-2}, 2^{k+2}]}^x \chi \right].$$

Since χ is rapidly decaying, it then follows that, repeating the estimates from (6.166),

(6.168)

$$\begin{aligned}
& \left\| \Delta_0^x \int_{\mathbb{R}^D} f(x, v) \chi(x) \phi(v) dv \right\|_{L_x^p} \\
& \leq C \left\| \int_{\mathbb{R}^D} \left[\Delta_0^x f S_4^x \chi + \Delta_1^x f S_8^x \chi + \Delta_2^x f S_{16}^x \chi + \sum_{k=2}^{\infty} \Delta_{2^k}^x f \Delta_{[2^{k-2}, 2^{k+2}]}^x \chi \right] \phi(v) dv \right\|_{L_x^p} \\
& \leq C \left\| \int_{\mathbb{R}^D} \Delta_0^x f(x, v) \phi(v) dv \right\|_{L_x^{r_0}} + C \sum_{k=0}^{\infty} 2^{ka} \left\| \int_{\mathbb{R}^D} \Delta_{2^k}^x f(x, v) \phi(v) dv \right\|_{L_x^{r_0}} \\
& \leq C \|f\|_{B_{r_0, p_0, q_0}^{a, \alpha}(\mathbb{R}_x^D \times \mathbb{R}_v^D)} \|\phi\|_{B_{p'_0, q'_0}^{-\alpha}(\mathbb{R}_v^D)},
\end{aligned}$$

which concludes the proof of the theorem. \square

APPENDIX A. SOME PARADIFFERENTIAL CALCULUS

For the sake of completeness and convenience of the reader, we include below a technical lemma on basic paradifferential calculus.

Lemma A.1. *Let $1 \leq p, q, r \leq \infty$, $s \in \mathbb{R}$ and $\phi(v) \in \mathcal{S}(\mathbb{R}^D)$. Then, for any $f(x, v) \in \tilde{L}^r(\mathbb{R}_x^D; B_{p, q}^s(\mathbb{R}_v^D))$, it holds that*

$$(A.1) \quad \|f\phi\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p, q}^s(\mathbb{R}_v^D))} \leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p, q}^s(\mathbb{R}_v^D))},$$

where $C > 0$ only depends on ϕ and on fixed parameters.

Proof. We begin formally with the standard Bony's decomposition (cf. [10])
(A.2)

$$f\phi = \left(\Delta_0^v f + \sum_{k=0}^{\infty} \Delta_{2^k}^v f \right) \left(\Delta_0^v \phi + \sum_{k=0}^{\infty} \Delta_{2^k}^v \phi \right) = T(f, \phi) + T(\phi, f) + R(f, \phi),$$

where we have denoted the paraproducts

$$(A.3) \quad T(f, \phi) = \sum_{k=2}^{\infty} \Delta_{2^k}^v f S_{2^{k-3}}^v \phi \quad \text{and} \quad T(\phi, f) = \sum_{k=2}^{\infty} \Delta_{2^k}^v \phi S_{2^{k-3}}^v f,$$

and the remainder

$$(A.4) \quad R(f, \phi) = \Delta_0^v f R_0^v \phi + \sum_{k=0}^{\infty} \Delta_{2^k}^v f R_{2^k}^v \phi,$$

where $R_0^v \phi = \Delta_0^v \phi + \sum_{j=0}^1 \Delta_{2^j}^v \phi$, $R_1^v \phi = \Delta_0^v \phi + \sum_{j=0}^2 \Delta_{2^j}^v \phi$,

$$R_2^v \phi = \Delta_0^v \phi + \sum_{j=0}^3 \Delta_{2^j}^v \phi \quad \text{and} \quad R_{2^k}^v \phi = \sum_{j=-2}^2 \Delta_{2^{k+j}}^v \phi \quad \text{if } k \geq 2.$$

We then estimate $T(f, \phi)$, $T(\phi, f)$ and $R(f, \phi)$ separately.

The control of the paraproduct $T(f, \phi)$ proceeds as follows. First, notice that the support of the Fourier transform in the velocity variable of $S_{2^{k-3}}^v \phi$ is contained inside a closed ball of radius 2^{k-2} centered at the origin and is thus separated from the support of the Fourier transform of $\Delta_{2^k}^v f$ by a distance of at least 2^{k-2} . Therefore, the Fourier transform of the general term in the sum of the paraproduct is supported inside an annulus centered at the origin of inner radius 2^{k-2} and outer radius $9 \cdot 2^{k-2}$. As a consequence, we have that

$$(A.5) \quad \begin{aligned} \Delta_0^v \left[\Delta_{2^j}^v f S_{2^{j-3}}^v \phi \right] &\equiv 0 \quad \text{for all } j \geq 2 \\ \text{and } \Delta_{2^k}^v \left[\Delta_{2^j}^v f S_{2^{j-3}}^v \phi \right] &\equiv 0 \quad \text{if } |j - k| \geq 3. \end{aligned}$$

We may then proceed with the estimation

$$(A.6) \quad \begin{aligned} \|T(f, \phi)\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))} &= \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v T(f, \phi)\|_{L_x^r L_v^p} \right\}_{k=0}^{\infty} \right\|_{\ell^q} \\ &\leq \left\| \left\{ \sum_{\substack{j=2 \\ |j-k|\leq 2}}^{\infty} 2^{(k-j)s} 2^{js} \|\Delta_{2^j}^v f S_{2^{j-3}}^v \phi\|_{L_x^r L_v^p} \right\}_{k=0}^{\infty} \right\|_{\ell^q} \\ &\leq C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v f S_{2^{k-3}}^v \phi\|_{L_x^r L_v^p} \right\}_{k=2}^{\infty} \right\|_{\ell^q} \\ &\leq C \|\phi\|_{L_v^\infty} \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v f\|_{L_x^r L_v^p} \right\}_{k=2}^{\infty} \right\|_{\ell^q} \\ &\leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))}. \end{aligned}$$

Regarding the paraproduct $T(\phi, f)$, we handle it through the following similar calculation, using that $\phi(v)$ is rapidly decaying,

$$\begin{aligned}
& \|T(\phi, f)\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))} \\
& \leq C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v \phi S_{2^{k-3}}^v f\|_{L_x^r L_v^p} \right\}_{k=2}^\infty \right\|_{\ell^q} \\
& \leq C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v \phi\|_{L_v^\infty} \|S_{2^{k-3}}^v f\|_{L_x^r L_v^p} \right\}_{k=2}^\infty \right\|_{\ell^q} \\
& \leq C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v \phi\|_{L_v^\infty} \right\}_{k=2}^\infty \right\|_{\ell^q} \|\Delta_0^v f\|_{L_x^r L_v^p} \\
& + \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v \phi\|_{L_v^\infty} \sum_{j=0}^{k-3} 2^{-j(s \wedge 0)} 2^{j(s \wedge 0)} \|\Delta_{2^j}^v f\|_{L_x^r L_v^p} \right\}_{k=3}^\infty \right\|_{\ell^q} \\
& \leq C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v \phi\|_{L_v^\infty} \right\}_{k=2}^\infty \right\|_{\ell^q} \|\Delta_0^v f\|_{L_x^r L_v^p} \\
& + \left\| \left\{ 2^{k(s+1)-k(s \wedge 0)} \|\Delta_{2^k}^v \phi\|_{L_v^\infty} \right\}_{k=3}^\infty \right\|_{\ell^q} \left\| \left\{ 2^{k(s \wedge 0)} \|\Delta_{2^k}^v f\|_{L_x^r L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
& \leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^{s \wedge 0}(\mathbb{R}_v^D))}.
\end{aligned} \tag{A.7}$$

Regarding the remainder, we first notice that the Fourier transforms of $\Delta_0^v f R_0^v \phi$ and $\Delta_{2^k}^v f R_{2^k}^v \phi$, for all $k \in \mathbb{N}$, are supported inside closed balls centered at the origin of respective radii 5 and $10 \cdot 2^k$. It follows that

$$\begin{aligned}
& \Delta_{2^k}^v [\Delta_0^v f R_0^v \phi] \equiv 0 \quad \text{if } k \geq 4 \\
& \text{and } \Delta_{2^k}^v [\Delta_{2^j}^v f R_{2^j}^v \phi] \equiv 0 \quad \text{if } k \geq j + 5.
\end{aligned} \tag{A.8}$$

Therefore, using once again that $\phi \in \mathcal{S}(\mathbb{R}^D)$,

$$\begin{aligned}
& \|R(f, \phi)\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))} \\
& \leq \|\Delta_0^v R(f, \phi)\|_{L_x^r L_v^p} + \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v R(f, \phi)\|_{L_x^r L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \\
& \leq C \left(\|\Delta_0^v f R_0^v \phi\|_{L_x^r L_v^p} + \sum_{k=0}^\infty \|\Delta_{2^k}^v f R_{2^k}^v \phi\|_{L_x^r L_v^p} \right) \\
& + C \left\| \left\{ 2^{ks} \left\| \sum_{j=k-4}^\infty \Delta_{2^j}^v f R_{2^j}^v \phi \right\|_{L_x^r L_v^p} \right\}_{k=4}^\infty \right\|_{\ell^q} \\
& \leq C \left(\|\Delta_0^v f\|_{L_x^r L_v^p} \|R_0^v \phi\|_{L_v^\infty} + \sum_{k=0}^\infty \|\Delta_{2^k}^v f\|_{L_x^r L_v^p} \|R_{2^k}^v \phi\|_{L_v^\infty} \right) \\
& + C \sum_{j=0}^\infty \left\| \Delta_{2^j}^v f \right\|_{L_x^r L_v^p} \left\| R_{2^j}^v \phi \right\|_{L_v^\infty} \left\| \left\{ 2^{ks} \right\}_{k=4}^{j+4} \right\|_{\ell^q} \\
& \leq C \|\Delta_0^v f\|_{L_x^r L_v^p} \|R_0^v \phi\|_{L_v^\infty} \\
& + C \left\| \left\{ 2^{ks} \|\Delta_{2^k}^v f\|_{L_x^r L_v^p} \right\}_{k=0}^\infty \right\|_{\ell^q} \left\| \left\{ 2^{k-k(s \wedge 0)} \|R_{2^k}^v \phi\|_{L_v^\infty} \right\}_{k=0}^\infty \right\|_{\ell^{q'}} \\
& \leq C \|f\|_{\tilde{L}^r(\mathbb{R}_x^D; B_{p,q}^s(\mathbb{R}_v^D))}.
\end{aligned} \tag{A.9}$$

Finally, incorporating (A.6), (A.7) and (A.9) into the decomposition (A.2), we easily deduce that the estimate (A.1) holds true, which concludes the demonstration. \square

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